ON THE ABEL-JACOBI MAP FOR DIVISORS OF HIGHER RANK ON A CURVE

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1. Introduction

The aim of this paper is to present an algebro-geometric approach to the study of the geometry of the moduli space of stable bundles on a smooth projective curve defined over an algebraically closed field k, of arbitrary characteristic. This establishes a bridge between the arithmetic approach of [15,17,9] and the gauge group approach of [2]. It might also help explain some of the mysterious analogies observed by Atiyah and Bott, in the form of "Weil conjectures" for the infinite dimensional algebraic variety associated to the ind-variety considered below.

One of the basic ideas is to consider a notion of divisor of higher rank and a suitable Abel-Jacobi map generalizing the classical notions in rank one. Let $\mathcal{O}_C$ be the structure sheaf of the curve C and let K be its field of rational functions, considered as a constant $\mathcal{O}_C$-module.

We define a divisor of rank r and degree n, an (r,n)-divisor for short, to be any coherent sub $\mathcal{O}_C$-module of $K^r = K^{\oplus r}$ having rank r and degree n. Since C is smooth, these submodules are locally free and coincide with the matrix divisors defined by A. Weil [31, 13].

Denote by $\mathrm{Div}^{r,n}_{C/k}$ the set of all (r,n)-divisors. This set can be identified with the set of rational points of an algebraic ind-variety $\mathbf{Div}^{r,n}_{C/k}$ that may be described as follows.

For any effective ordinary divisor D set

$$\mathrm{Div}^{r,n}_{C/k}(D) = \{E \in \mathrm{Div}^{r,n}_{C/k} | E \subseteq \mathcal{O}_C(D)^r\}$$

where $\mathcal{O}_C(D)^r$ is considered as a sub $\mathcal{O}_C$-module of K^r. Clearly

$$\mathrm{Div}^{r,n}_{C/k} = \bigcup_{D \geq 0} \mathrm{Div}^{r,n}_{C/k}(D)$$

and the elements of the set $\mathrm{Div}^{r,n}_{C/k}(D)$ can be identified with the rational points of the scheme $\mathrm{Quot}^m_{\mathcal{O}_C(D)^r/X/k}$, $m = r \cdot \deg D - n$, parametrizing torsion quotients of $\mathcal{O}_C(D)^r$ having degree m [14]. These are smooth projective varieties and for any pair of effective divisors D, D' with $D \leq D'$ the inclusion

$$\mathrm{Div}^{r,n}_{C/k}(D) \to \mathrm{Div}^{r,n}_{C/k}(D')$$

is induced by a closed immersion of the corresponding varieties.

It is natural to stratify the ind-variety $\mathbf{Div}^{r,n}_{C/k}$ according to Harder-Narasimhan type

$$\mathbf{Div}^{r,n}_{C/k} = (\mathbf{Div}^{r,n}_{C/k})^{ss} \cup \bigcup_{P \neq ss} \mathbf{S}_P$$

where $(\mathbf{Div}^{r,n}_{C/k})^{ss}$ is the open ind-subvariety of semistable divisors. The cohomology of each stratum stabilizes and this stratification is perfect. (Here cohomology means ℓ-adic cohomology for a suitable prime ℓ.) In particular, there is an identity of Poincaré series

$$P(\mathbf{Div}^{r,n}_{C/k}; t) = P((\mathbf{Div}^{r,n}_{C/k})^{ss}; t) \; + \sum_{P \neq ss} P(\mathbf{S}_P; t) \cdot t^{2\,\mathrm{d}_P} \; . \tag{1.1}$$

where d_P is the codimension of $\mathbf{S}_P$ (see Proposition 5.2 below for an explicit expression.)

Let r and n be coprime. Then the notions of stable and semistable bundle over C coincide, and the moduli space $N(r,n)$ of stable vector bundles having rank r and degree n is in this case a smooth projective algebraic variety. It is natural to define, by analogy with the classical case, Abel-Jacobi maps

$$\boldsymbol{\vartheta} : (\mathbf{Div}^{r,n}_{C/k})^{ss} \longrightarrow N(r,n) \; ,$$

by assigning to a divisor E its isomorphism class as a vector bundle. There is a "coherent locally free" module $\boldsymbol{E}$ over $N(r,n)$, considered as a constant ind-variety, and a morphism

$$\begin{tikzcd} (\mathbf{Div}^{r,n}_{C/k})^{ss} \arrow[rr, "\mathbf{i}"] \arrow[dr, "\boldsymbol{\vartheta}"'] & & \mathbb{P}(\boldsymbol{E}) \arrow[dl] \\ & N(r,n) & \end{tikzcd}$$

inducing an isomorphism in cohomology. It follows that

$$H^*((\mathbf{Div}^{r,n}_{C/k})^{ss}) = H^*(N(r,n))[x]$$

where x is an independent variable of degree 2, and hence

$$P(N(r,n); t) = (1 - t^2) \cdot P((\mathbf{Div}^{r,n}_{C/k})^{ss}; t) \; . \tag{1.2}$$

If $(r_1', d_1'), \ldots, (r_l', d_l')$ is the type determined by P, then the map

$$\boldsymbol{\delta} : \; (\mathbf{Div}^{r_1', d_1'}_{C/k})^{ss} \times \cdots \times (\mathbf{Div}^{r_l', d_l'}_{C/k})^{ss} \longrightarrow \mathbf{S}_P \; ,$$

given by $(E_1, \ldots, E_l) \mapsto E_1 \oplus \cdots \oplus E_l$, induces an isomorphism in cohomology and one has

$$P(\mathbf{S}_P; t) = \prod_{1 \leq j \leq l} P((\mathbf{Div}^{r_j', d_j'}_{C/k})^{ss}; t) \; . \tag{1.3}$$

In order to find the Betti numbers of $N(r,n)$ it suffices, by (1.2), to know those of

The varieties $\mathrm{Div}^{r,n}_{C/k}(D)$ are analogous to Grassmannians and share with them the property of having a decomposition into Schubert "strata" which may be defined, for example, in terms of a toric action. It follows easily [3] that their cohomology is free of torsion and that their Betti numbers are given by

$$P(\mathrm{Div}^{r,n}_{C/k}(D);t) = \sum_{\mathbf{m}} t^{2d_{\mathbf{m}}}\ P(C^{(m_1)};t)\cdots P(C^{(m_r)};t)$$

where $\mathbf{m} = (m_1, \ldots, m_r)$ is any partition of $m = r \cdot \deg D - n$ by nonnegative integers, $d_{\mathbf{m}} = \sum_{1\le i\le r}(i-1)m_i$, and $C^{(m)}$ stands for the symmetric product. This reduces the calculation, in a sense, to the well-known abelian case $r = 1$, and since the cohomology of $\mathbf{Div}^{r,n}_{C/k}$ stabilizes, one obtains

$$P(\mathbf{Div}^{r,n}_{C/k};t) = \frac{\prod_{j=1}^{r}(1+t^{2j-1})^{2g}}{(1-t^{2r})\prod_{j=1}^{r-1}(1-t^{2j})^2}\ . \tag{1.4}$$

There is a clear formal analogy between this approach and that of [2]. In [2] Atiyah and Bott consider the space $\mathcal{C}(r,n)$ of holomorphic structures on a fixed C^∞ vector bundle E together with the action on it of the complexified gauge group $\mathcal{G} = \mathrm{Aut}\ (E)$. They stratify $\mathcal{C}(r,n)$ according to Harder-Narasimhan type and, using $\mathcal{G}$-equivariant cohomology, obtain formulæ, similar to the ones above, that reduce the calculation of $P(N(r,n);t)$ to that of $P(B\mathcal{G};t)$ for arbitrary r. Since the Poincaré series of the classifying space $B\mathcal{G}$ coincides with the expression (1.4), the two approaches give the same formulæ for the Betti numbers as in [24,15,17,9,30,18,6].

If one considers, for k the algebraic closure of a finite field $\mathbb{F}_q$, the Abel-Jacobi map from the set of $\mathbb{F}_q$-rational points of $\mathbf{Div}^{r,n}_{C/k}$ to the (finite) set of isomorphism classes of $\mathbb{F}_q$-rational bundles, having rank r and degree n, then by simple considerations of cardinalities one can prove the function-field analogue of the Siegel formula [11]. Recall that this formula, equivalent to the Tamagawa number of $\mathrm{SL}(r,K)$ being 1, is the keystone for the computation of the Betti numbers of $N(r,n)$ in the arithmetic approach, via the Weil conjectures, of Harder, Narasimhan, Desale and Ramanan [15,17,9].

2. Some notation and preliminary remarks

Let k be an algebraically closed field of arbitrary characteristic. In this paper, an ind-variety $\mathbf{X} = \{X_\lambda, f_{\lambda\mu}\}_{\lambda,\mu\in\Lambda}$ is simply an inductive system of k-algebraic varieties indexed by some filtered ordered set Λ. The indexing set Λ will often be the set of effective divisors on the curve C. We shall say that $\mathbf{X}$ is smooth if there is a $\lambda_0 \in \Lambda$ such that for every $\lambda \ge \lambda_0$, X_λ is smooth.

Given another ind-variety $\mathbf{Y} = \{Y_\rho,\ g_{\rho\sigma}\}_{\rho,\sigma\in\Gamma}$, a morphism $\mathbf{\Phi} = \{\alpha,\ \{\phi_\lambda\}_{\lambda\in\Lambda}\}$ from $\mathbf{X}$ to $\mathbf{Y}$ consists of an order preserving map $\alpha : \Lambda \longrightarrow \Gamma$ together with a family of morphisms

$$\phi_\lambda : X_\lambda \to Y_{\alpha(\lambda)}$$

satisfying the obvious commutativity properties. We shall say that a morphism $\mathbf{\Phi}$

(2) for every integer k there exists a $\lambda_k \in \Lambda$ such that for every $\lambda \geq \lambda_k$ the map $\phi_\lambda : X_\lambda \to Y_{\alpha(\lambda)}$ is an open immersion and the codimension of $Y_{\alpha(\lambda)} - \phi_\lambda(X_\lambda)$ in $Y_{\alpha(\lambda)}$ is greater than k.

Unless otherwise stated cohomology will mean ℓ-adic cohomology

$$H^*(X) = \varprojlim_r H^*(X_{\text{ét}}, \mathbb{Z}/\ell^r\mathbb{Z})$$

for a suitable prime ℓ distinct from the characteristic of the field and coprime with the ranks and the degrees of the vector bundles considered. If the base field is $\mathbb{C}$, then cohomology could also be singular cohomology with coefficients in $\mathbb{Z}$ or in any field (with similar restrictions on the characteristic.)

Whenever we mention the Künneth formula, we are referring to the one, without supports, in [8].

We say that the cohomology of the ind-variety $\mathbf{X}$ stabilizes if for every integer n there exists a λ_n such that for every λ, μ with $\mu \geq \lambda \geq \lambda_n$ and for every j with $j \leq n$ the map

$$(f_{\lambda\mu})^* : H^j(X_\mu) \longrightarrow H^j(X_\lambda)$$

is an isomorphism. The cohomology ring of $\mathbf{X}$ will be the projective limit

$$H^*(X) = \varprojlim_\lambda H^*(X_\lambda) \ .$$

We may also refer to the ring $H^*(\mathbf{X})$ as the stable cohomology ring of $\mathbf{X}$.

The Poincaré series of $\mathbf{X}$ is by definition that of $H^*(\mathbf{X})$.

Note that if $\boldsymbol{\Phi} : \mathbf{X} \to \mathbf{Y}$ is a quasi-isomorphism of smooth ind-varieties, then $\boldsymbol{\Phi}^* : H^*(\mathbf{Y}) \to H^*(\mathbf{X})$ is an isomorphism (see, for example, Lemma 9.1 in Chapter VI of [23].) Furthermore, the cohomology of $\mathbf{X}$ stabilizes if, and only if, that of $\mathbf{Y}$ does.

An $\mathcal{O}_{\mathbf{X}}$-module will be, in this paper, a family $\boldsymbol{E} = \{\mathcal{E}_\lambda,\ \eta_{\lambda,\mu}\}_{\lambda,\mu\in\Lambda}$ where $\mathcal{E}_\lambda$ is a $\mathcal{O}_{X_\lambda}$-module and

$$\eta_{\lambda,\mu} : f^*_{\lambda,\mu}\mathcal{E}_\mu \longrightarrow \mathcal{E}_\lambda$$

are surjective homomorphisms (for example isomorphisms.) To such a module, with coherent $\mathcal{E}_\lambda$'s say, one can associate (as in E.G.A.) an ind-scheme

$$\mathbb{P}(\boldsymbol{E}) \longrightarrow \mathbf{X}$$

and a fundamental invertible module $\mathcal{O}_{\mathbb{P}}(1)$ over it.

Two special situations will often arise. The first one, is that of a "coherent locally free" module of "infinite rank" over a "finite dimensional" base $\mathbf{X}$. Here "coherent locally free" means that the coherent modules $\mathcal{E}_\lambda$ are locally free for λ in a cofinal subset, "infinite rank" that the ranks of the $\mathcal{E}_\lambda$ are unbounded, and "finite dimensional" that the cohomology of $\mathbf{X}$ stabilizes and, furthermore, vanishes beyond a certain dimension. In this case it is easy to show that

$$H^*(\mathbb{P}(\boldsymbol{E})) = H^*(\mathbf{X})[\xi]$$

where ξ, the Chern class of $\mathcal{O}_{\mathbb{P}}(1)$, is algebraically independent. The second one, is that of a "coherent locally free" module of "finite rank" (i.e. its components

3. The stratification associated to a direct sum decomposition

Let X be a projective variety over a field k, and let $\mathcal{F}$ be any coherent $\mathcal{O}_X$-module. Recall that there is a universal family of coherent quotients of $\mathcal{F}$ parametrized by a scheme $Q = \mathrm{Quot}_{\mathcal{F}/X/k}$ which is a union of projective schemes [14]. Since there is an obvious bijection between coherent quotients of $\mathcal{F}$ and coherent submodules of $\mathcal{F}$, we can also consider Q as the variety parametrizing the universal family of coherent submodules of $\mathcal{F}$. We shall do this systematically throughout the paper.

Recall, that if $E \subseteq \mathcal{F}$ corresponds to a point of Q then the Zariski tangent space to Q at E can be identified with $\mathrm{Hom}(E, \mathcal{F}/E)$. If $\mathrm{Ext}^1(E, \mathcal{F}/E) = 0$, then Q is smooth at E.

It is often useful to think of the schemes $\mathrm{Quot}_{\mathcal{F}/X/S}$ as a generalization of Grassmannians (which correspond to the case $X = S$.) One of the key techniques for studying the geometry of Grassmannians is their decomposition into Schubert cells. The classical way of defining these cells is via incidence conditions. There is, however, a different approach via flows under the action of a suitable torus which yields the same results. From this point of view, one should think of the results in [3] as providing generalized Schubert "strata" in terms of a torus action.

Note that associated to any direct sum decomposition

$$\mathcal{F} = \mathcal{F}_1 \oplus \cdots \oplus \mathcal{F}_n$$

of a coherent module $\mathcal{F}$, there is a natural action of $\mathbb{G}_m^n$ on $\mathrm{Quot}_{\mathcal{F}/X/k}$ having

$$\mathrm{Quot}_{\mathcal{F}_1/X/k} \times_k \cdots \times_k \mathrm{Quot}_{\mathcal{F}_n/X/k}$$

as fixed points (proof as in [3].)

For simplicity, we shall restrict ourselves to the case $X = C$ and $n = 2$, but one could develop the general case similarly along the lines of [3] (see a paper, in preparation, by one of the authors.)

Choose $\lambda_1, \lambda_2 \in \mathbb{Z}$, with $\lambda_1 > \lambda_2$, and define a one-parameter subgroup $\lambda : \mathbb{G}_m \to \mathbb{G}_m^2$ by $\lambda(t) = (t^{\lambda_1}, t^{\lambda_2})$. The induced $\mathbb{G}_m$-action has the same fixed points as the original one, and we now have a decomposition of $\mathrm{Quot}_{\mathcal{F}/X/k}$ into strata, determined by the flow towards the different components of the fixed points.

It is possible, however, to describe these strata in terms closer to the classical description as follows.

Let $\mathcal{F}_1, \mathcal{F}_2$ be coherent $\mathcal{O}_C$-modules on the projective curve C. A coherent submodule E of $\mathcal{F} = \mathcal{F}_1 \oplus \mathcal{F}_2$ determines submodules E_1 and E_2 of $\mathcal{F}_1$ and $\mathcal{F}_2$, respectively, which make exact the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & \mathcal{F}_1 & \to & \mathcal{F}_1 \oplus \mathcal{F}_2 & \to & \mathcal{F}_2 & \to & 0 \\
 & & \cup & & \cup & & \cup & & \\
0 & \to & E_1 & \to & E & \to & E_2 & \to & 0
\end{array}$$

Here, the horizontal maps are the obvious ones.

Furthermore, the map $E \subset \mathcal{F}_1 \oplus \mathcal{F}_2 \to \mathcal{F}_1$ induces, after taking quotients with E_1, a homomorphism $\varphi : E_2 \to \mathcal{F}_1/E_1$. Conversely, given $E_1 \subseteq \mathcal{F}_1$, $E_2 \subseteq \mathcal{F}_2$ and $\varphi : E_2 \to \mathcal{F}_1/E_1$, we can recover E as the fibre product $\mathcal{F}_1 \times_{\mathcal{F}_1/E_1} E_2$.

above. Note that the associated modules E_1, E_2 do not change as E describes a $\mathbb{G}_m$-orbit in Q. Note also that, while this happens, φ is being multiplied by $t^{\lambda_1-\lambda_2}$. It is now clear that the fixed points under this action are the submodules of type $E = E_1 \oplus E_2$ with $E_1 \subseteq \mathcal{F}_1$ and $E_2 \subseteq \mathcal{F}_2$, and that one of the two stratifications of Q determined by the $\mathbb{G}_m$-action can be described as

$$Q^{(r,n)} = \bigcup_{\substack{r_1+r_2=r \\ n_1+n_2=n}} Q^{(r_1,n_1;r_2,n_2)}$$

where $Q^{(r_1,n_1;r_2,n_2)}$ consists of those E such that the associated modules E_1, E_2 have ranks r_1, r_2 and degrees n_1, n_2 respectively. Note that the other stratification is obtained after permutation of $\mathcal{F}_1$ with $\mathcal{F}_2$.

For $i = 1, 2$, let $Q_i^{(r_i,n_i)}$ be the scheme parametrizing (r_i, n_i)-submodules of $\mathcal{F}_i$. There is an obvious map

$$Q^{(r_1,n_1;r_2,n_2)} \to Q_1^{(r_1,n_1)} \times Q_2^{(r_2,n_2)}$$

whose fibre over (E_1, E_2) can be identified with the vector space Hom $(E_2, \mathcal{F}_1/E_1)$. If the dimension of the fibres is constant (e.g. if $\operatorname{Ext}^1(E_2, \mathcal{F}_1/E_1) = 0$ for every $(E_1, E_2) \in Q_1^{(r_1,n_1)} \times Q_2^{(r_2,n_2)}$), then $Q^{(r_1,n_1;r_2,n_2)}$ can be considered as a vector bundle over $Q_1^{(r_1,n_1)} \times Q_2^{(r_2,n_2)}$.

We shall refer to the stratification just described as being associated to the direct sum decomposition $\mathcal{F} = \mathcal{F}_1 \oplus \mathcal{F}_2$.

4. The ind-variety of divisors and its cohomology

Let C be a smooth projective curve of genus g, defined over an algebraically closed field k, and let $\mathcal{O}_C$ be its structure sheaf. We denote by K the constant $\mathcal{O}_C$-module determined by the function field of C.

Definition 4.1. A *divisor* of rank r and degree n over C, an (r, n)-divisor for short, is a coherent sub $\mathcal{O}_C$-module of K^r having degree n and rank r. We denote the set of all such divisors by $\mathrm{Div}^{r,n}_{C/k}$.

Note that, since C is smooth, coherent submodules of K^r are locally free and these divisors coincide with the matrix divisors defined by A. Weil [31, 13].

For any effective ordinary divisor D we have $\mathcal{O}_C^r \subseteq \mathcal{O}_C(D)^r \subseteq K^r$ and we define

$$\mathrm{Div}^{r,n}_{C/k}(D) = \{E | E \in \mathrm{Div}^{r,n}_{C/k} \text{ and } E \subseteq \mathcal{O}_C(D)^r)\} .$$

On each $\mathrm{Div}^{r,n}_{C/k}(D)$ there are natural structures of smooth projective algebraic variety. In fact, the elements of $\mathrm{Div}^{r,n}_{C/k}(D)$ can be identified with the rational points of the scheme $\mathrm{Quot}^m_{\mathcal{O}_C(D)^r/X/k}$, $m = r \cdot \deg D - n$, parametrizing torsion quotients of $\mathcal{O}_C(D)^r$ having degree m [14]. These are smooth projective varieties and for any pair of effective divisors D, D' with $D \le D'$ the inclusion

$$\mathrm{Div}^{r,n}_{C/k}(D) \to \mathrm{Div}^{r,n}_{C/k}(D')$$

The ind-variety $\mathbf{Div}^{r,n}_{C/k}$ is determined by the inductive system consisting of the varieties $\mathrm{Div}^{r,n}_{C/k}(D)$ and the closed immersions above.

Remark. One could also define, as in [4], an ind-variety $\mathbf{Q}^{r,n}$ given by

$$Q^{r,n}(D) = \mathrm{Quot}^{n+r\cdot\deg D}_{\mathcal{O}^r_C/X/k}$$

for every ordinary effective divisor D, and having as structure maps the morphisms

$$Q^{r,n}(D_1) \longrightarrow Q^{r,n}(D_2)$$

obtained by tensoring the submodules with $\mathcal{O}_C(-D)$, where $D = D_2 - D_1 \geq 0$. It is easy to show that there is a natural isomorphism of inductive systems

$$\mathbf{Div}^{r,n}_{C/k} \longrightarrow \mathbf{Q}^{r,n} \ ,$$

defined by tensoring the submodules with $\mathcal{O}_C(-D)$ for every ordinary effective divisor D.

We are now ready to state the following

Proposition 4.2. *The cohomology of* $\mathbf{Div}^{r,n}_{C/k}$ *is free of torsion, stabilizes and its Poincaré series is given by*

$$P(\mathbf{Div}^{r,n}_{C/k};t) = \frac{\prod_{j=1}^{r}(1+t^{2j-1})^{2g}}{(1-t^{2r})\prod_{j=1}^{r-1}(1-t^{2j})^2} \ .$$

Proof. We shall prove below that the cohomology stabilizes.

Recall ([3]; cf. also the remark at the end of this section) that the cohomology of $\mathrm{Div}^{r,n}_{C/k}(D)$ is free of torsion and that its Poincaré series is given by

$$P(\mathrm{Div}^{r,n}_{C/k}(D);t) = \sum_{\mathbf{m}} t^{2d_{\mathbf{m}}}\ P(C^{(m_1)};t)\cdots P(C^{(m_r)};t)$$

where $\mathbf{m} = (m_1,\dots,m_r)$ is any partition of $m = r\cdot\deg D - n$ by non-negative integers and $d_{\mathbf{m}} = \sum_{1\leq i\leq r}(i-1)m_i$.

Consider the formal power series $E(t,u)$ defined by

$$E(t,u) = \sum_{m\geq 0}\left(\sum_{|\mathbf{m}|=m} P(C^{(m_1)},t)\cdots P(C^{(m_r)},t)\cdot t^{2d_{\mathbf{m}}}\right)\cdot u^m$$

where $|\mathbf{m}| = \sum m_i$.

We know, see [21], that

$$\sum_{j\geq 0} P(C^{(j)},t)u^j = \frac{(1+ut)^{2g}}{(1-u)(1-ut^2)} \ .$$

Therefore

$$E(t,u) = \prod_{j=0}^{r-1}\frac{(1+u\ t^{2j+1})^{2g}}{(1-u\ t^{2j})(1-u\ t^{2j+2})} \ .$$

is a rational function, and

$$P(\mathbf{Div}^{r,n}_{C/k};t) = -\operatorname*{Res}_{u=1} E(t,u) = \frac{\prod_{j=1}^{r}(1+t^{2j-1})^{2g}}{(1-t^{2r})\prod_{j=1}^{r-1}(1-t^{2j})^2}$$

has the expected value.

Proposition 4.3. *If $D' \geq D$, then the morphism*

$$H^i(\mathrm{Div}^{r,n}_{C/k}(D') \longrightarrow H^i(\mathrm{Div}^{r,n}_{C/k}(D))$$

induced by the closed immersion $\mathrm{Div}^{r,n}_{C/k}(D) \to \mathrm{Div}^{r,n}_{C/k}(D')$ is an isomorphism for $0 \leq i < r \deg D - n$.

This proposition is an immediate consequence of the next lemma, since we can always factor the map $\mathrm{Div}^{r,n}_{C/k}(D) \to \mathrm{Div}^{r,n}_{C/k}(D')$ through a succession of maps of the type considered there.

Lemma 4.4. *Let $\mathcal{L}$, $\mathcal{F}$ be locally free $\mathcal{O}_C$-modules having ranks 1 and $r-1$ respectively. Fix a point P in C and let Q, Q' be the varieties of (r,n)-submodules of $\mathcal{L} \oplus \mathcal{F}$ and $\mathcal{L}(P) \oplus \mathcal{F}$ respectively. Then the morphism*

$$H^i(Q') \longrightarrow H^i(Q),$$

induced by the closed immersion $Q \to Q'$, is an isomorphism for $0 \leq i < \deg \mathcal{F} + \deg \mathcal{L} - n$.

Proof. Suppose first that $r = 1$. We can take $\mathcal{L} = \mathcal{O}_C(D)$ for some divisor D. Every submodule of $\mathcal{O}_C(D)$ is equal to $\mathcal{O}_C(F)$ for some $F \leq D$. The map $\mathcal{O}_C(F) \mapsto D - F$ gives an isomorphism $Q \longrightarrow C^{(m)}$, where $m = \deg \mathcal{L} - n$. Similarly, there is an isomorphism $Q' \longrightarrow C^{(m+1)}$ and a commutative diagram

$$\begin{array}{ccc} Q & \longrightarrow & C^{(m)} \\ \big\downarrow & & \big\downarrow \\ Q' & \longrightarrow & C^{(m+1)} \end{array}$$

where the map $C^{(m)} \to C^{(m+1)}$ is given by $H \mapsto H + P$. As is well known [21], the induced map

$$H^i(C^{(m+1)}) \longrightarrow H^i(C^{(m)})$$

is an isomorphism for $0 \leq i < m$, and a monomorphism for $i = m$. Therefore, the same holds for the map $H^i(Q') \longrightarrow H^i(Q)$.

In general, consider the stratification of Q associated to the direct sum $\mathcal{L} \oplus \mathcal{F}$. We can write

$$Q = Q_0 \cup Q_1 \cup Q_2 \cup \dots$$

where Q_j consists of the submodules E of $\mathcal{L} \oplus \mathcal{F}$ which project onto a submodule E_2 of $\mathcal{F}$ having degree $\deg \mathcal{F} - j$. The subvariety Q_j has codimension j in Q and can be considered as a vector bundle over $V_j \times Z_j$, where V_j is the variety of submodules of $\mathcal{L}$ having rank 1 and degree $n - \deg \mathcal{F} + j$, and Z_j is the variety of submodules of $\mathcal{F}$ having rank $r-1$ and degree $\deg \mathcal{F} - j$.

Analogously, we have a stratification

$$Q' = Q'_0 \cup Q'_1 \cup Q'_2 \cup \dots$$

We have, for every j, a commutative diagram of Gysin exact sequences

$$\begin{array}{ccccccc}
\to H^{p-2j}(Q'_j) & \longrightarrow & H^p(Q'_0\cup\cdots\cup Q'_j) & \longrightarrow & H^p(Q'_0\cup\cdots\cup Q'_{j-1}) \to \\
\downarrow & & \downarrow & & \downarrow \\
\to H^{p-2j}(Q_j) & \longrightarrow & H^p(Q_0\cup\cdots\cup Q_j) & \longrightarrow & H^p(Q_0\cup\cdots\cup Q_{j-1}) \to
\end{array}$$

We know that the map $H^i(V'_j)\to H^i(V_j)$ is an isomorphism for $0\le i<\deg\mathcal{L}+\deg\mathcal{F}-n-j$ and a monomorphism for $i=\deg\mathcal{L}+\deg\mathcal{F}-n-j$. Using the isomorphisms

$$H^*(Q'_j)\simeq H^*(V'_j)\otimes H^*(Z_j)$$

and

$$H^*(Q_j)\simeq H^*(V_j)\otimes H^*(Z_j)$$

we see that $H^i(Q'_j)\to H^i(Q_j)$ is an isomorphism for $0\le i<\deg\mathcal{L}+\deg\mathcal{F}-n-j$ and a monomorphism for $i=\deg\mathcal{L}+\deg\mathcal{F}-n-j$.

At this point the five-lemma applied to the diagram above allows us to perform an induction on j and conclude the proof of the lemma.

Remark. . Since the stratification $Q=Q_0\cup Q_1\cup\dots$ comes from an action of $\mathbb{G}_m$, it is easy to show that it is perfect i.e. the bottom Gysin sequence appearing in the diagram above breaks up into short exact sequences, one for every j. It is now possible, by induction on r, to show that the cohomology of $\mathrm{Div}^{r,n}_{C/k}(D)$ is free of torsion, and to compute its Poincaré polynomial.

5. The Shatz stratification of the ind-variety of divisors

Let us recall some standard definitions. The slope $\mu(E)$ of a coherent $\mathcal{O}_C$-module E is defined by

$$\mu(E)=\frac{\deg E}{\text{rank } E}\ .$$

A coherent locally free $\mathcal{O}_C$-module E is called semistable (resp. stable) if for every proper submodule F of E one has

$$\mu(F)\le\mu(E)\qquad\qquad (\text{resp. } \mu(F)<\mu(E))\ .$$

Recall the following two properties of semistable modules.

(5.1) If E is semistable and $\mu(E)>2g-1$, then E is generated by its global sections and $H^1(C,E)=0$.

(5.2) If E and F are semistable, then Hom $(E,F)=0$ whenever $\mu(E)>\mu(F)$.

The Harder-Narasimhan filtration of a coherent locally free $\mathcal{O}_C$-module E is the unique filtration

$$0=E_0\subset E_1\subset\cdots\subset E_l=E$$

with $G_i=E_i/E_{i-1}$ semistable and

$$\mu(G_1)>\mu(G_2)>\cdots>\mu(G_l)\ .$$

Note that if $\{(r_i, d_i)\}_{0\leq i\leq l}$ is the polygon of E, then the type of E is given by $\{(r'_i, d'_i)\}_{1\leq i\leq l}$ with

$$(r'_i, d'_i) = (r_i - r_{i-1}, d_i - d_{i-1}).$$

We shall use primed letters for the type of a bundle, and unprimed letters for the vertices of its polygon.

Recall that the Shatz polygon P_E of E is characterized by the following property.

(5.3) If F is a (s, m)-submodule of E, then the point (s, m) lies either on P_E or below it.

Consider the set $\mathcal{P}_{r,n}$ of all strictly convex polygons of $\mathbb{R}^2$ joining $(0, 0)$ to (r, n). If P, and P' are in $\mathcal{P}_{r,n}$, then we say that $P \geq P'$ whenever P lies above P'.

Let $\mathcal{E}$ be a family of vector bundles of rank r and degree n over C parametrized by a scheme T. For $t \in T$ let P_t be the polygon of $\mathcal{E}_t$. For each $P \in \mathcal{P}_{r,n}$ we define subsets of T

$$F_P(T) = \{t \in T | P_t > P\} \quad , \quad \Omega_P(T) = T - F_P(T)$$

and

$$S_P(T) = \{t \in T | P_t = P\} \ .$$

We may also write $(T)^P$, instead of $S_P(T)$.

The partition

$$T = \bigcup_{P\in\mathcal{P}_{r,n}} S_P(T)$$

is called the Shatz stratification of T.

We summarize, in the next proposition, some general properties of this stratification in the case $T = \mathrm{Div}^{r,n}_{C/k}(D)$. In order to state them we shall need some notation.

Definition 5.1. Let X be a scheme and let

$$0 = \mathcal{F}_0 \subseteq \mathcal{F}_1 \cdots \subseteq \mathcal{F}_l = \mathcal{F}$$

be a filtration of an $\mathcal{O}_X$-module $\mathcal{F}$. We denote by $\mathcal{H}om_-(\mathcal{F}, \mathcal{F})$ the subsheaf of $\mathcal{H}om(\mathcal{F}, \mathcal{F})$ consisting of the germs of homomorphism preserving the filtration. We also set

$$\mathcal{H}om_+(\mathcal{F}, \mathcal{F}) = \mathcal{H}om(\mathcal{F}, \mathcal{F})/\mathcal{H}om_-(\mathcal{F}, \mathcal{F}) \ .$$

Proposition 5.2. *Let $T_D = \mathrm{Div}^{r,n}_{C/k}(D)$ and fix a Shatz polynomial*

$$P = \{(r_0, d_0), \ldots, (r_l, d_l)\} \ .$$

We set, as usual, $r'_i = r_i - r_{i-1}$ and $d'_i = d_i - d_{i-1}$. Then

1. *$F_P(T_D)$ is closed and $\Omega_P(T_D)$ is an open set;*
2. *$S_P(D) = S_P(T_D)$ is in a natural way a closed subscheme of $\Omega_P(T_D)$. Moreover, if $\mathcal{U}^P_D$ is the restriction to $S_P(D)$ of the universal family of divisors, then $\mathcal{U}^P_D$ has a universal Harder-Narasimhan filtration*

such that

$$\mathcal{U}_i^P/\mathcal{U}_{i-1}^P$$

is locally free for $i = 1, \ldots, l$.

(3) *If* $\deg D > \dfrac{d'_1}{r'_1} + 2g - 1$, *then* $S_P(D)$ *is smooth, has codimension*

$$d_P = \sum_{i>j} r'_i r'_j \left[\left(\frac{d'_j}{r'_j} - \frac{d'_i}{r'_i} \right) + g - 1 \right]$$

and its normal bundle can be identified with

$$R^1 q_* \, \mathcal{H}om_+(\mathcal{U}_D^P, \, \mathcal{U}_D^P)$$

where $q : C \times S_P(D) \longrightarrow S_P(D)$ *is the natural projection.*

(4) *If* $\deg D \leq \dfrac{d'_1}{r'_1} + 2g - 1$, *then*

$$\operatorname{codim} S_P(D) \geq \deg D - c$$

where c *is a constant independent of* P *and* D.

Proof. Most of the assertions made in the proposition are easy consequences of general results [29,30]; however, for the convenience of the reader, we outline a proof.

Note that throughout this outline, if E is a submodule of $\mathcal{O}_C(D)^r$, we shall often write $\widetilde{E}$ instead of $\mathcal{O}_C(D)^r/E$.

Let $\mathcal{D}_D^{s,m} = \mathcal{D}_r^{s,m}(D)$ be the projective scheme parametrizing the (s,m)-submodules of $\mathcal{O}_C(D)^r$. We shall show that the proposition holds, more generally, for these varieties.

Let P be any polygon with vertices

$$(0,0) = (s_0, m_0), \ldots, (s_l, m_l) = (s, m) \; .$$

We denote by Drap_D^P the projective scheme parametrizing the universal family of flags

$$0 = E_0 \subset E_1 \cdots \subset E_l \subset \mathcal{O}_C(D)^r \tag{5.4}$$

having P as associated polygon i.e.

$$(\text{rank } E_i, \deg E_i) = (s_i, m_i)$$

for $i = 0, \ldots, l$. The existence of Drap_D^P follows, by induction on l, from that of the schemes of quotients.

In fact Drap_D^P can be identified, in a natural way, with a closed subscheme of

$$\mathcal{D}_D^{s_1,m_1} \times \ldots \mathcal{D}_D^{s_l,m_l} \; .$$

The Zariski tangent space to Drap_D^P at the point corresponding to a flag like (5.4) can be identified with the subspace of

consisting of the l-tuples $(f_1, \ldots, f_l)$ such that f_{i+1} and f_i map to the same element in Hom $(E_i, \widetilde{E}_{i+1})$ under the obvious maps.

Let us begin the proof of (1). Consider the map

$$\pi_P : \mathrm{Drap}_D^P \longrightarrow \mathcal{D}_D^{s,m}$$

which assigns to a filtration, as in (5.4), the submodule E_l. Since π_P is projective, its image is closed in $\mathcal{D}_D^{s,m}$. Also, as a consequence of (5.3), we have for every (s,m)-polygon P

$$F_P(\mathcal{D}_D^{s,m}) = \bigcup_{P'>P} \mathrm{Im}\, \pi_{P'} \ .$$

Since the degree of any submodule of $\mathcal{O}_C(D)^r$ is bounded by $r \deg D$, there are only finitely many polygons $P' > P$ with nonempty $\mathrm{Drap}_D^{P'}$. This proves (1).

Let now $U = \Omega_P(\mathcal{D}_D^{s,m})$ and consider

$$\pi_P' : \pi_P^{-1}(U) \longrightarrow U \ ,$$

the restriction of π_P. This map is still projective, and its image coincides with $S_P(\mathcal{D}_D^{s,m})$ and is closed in U. The uniqueness of the Harder-Narasimhan filtration shows that π_P' is injective. We claim that it is actually a closed immersion. What needs to be shown (cf. [30]) is that if $(f_1, \ldots f_l)$ belongs to the tangent space to Drap_D^P at a filtration like (5.4) which also happens to be the Harder-Narasimhan filtration of E_l, then f_i is uniquely determined by f_{i+1} for $i = 1, \ldots, l-1$. But note that, as a consequence of (5.2), one has Hom $(E_i, E_{i+1}/E_i) = 0$; therefore the map

$$\mathrm{Hom}\ (E_i, \widetilde{E}_i) \rightarrow \mathrm{Hom}\ (E_i, \widetilde{E}_{i+1})$$

is injective and f_{i+1} uniquely determines f_i.

We can now identify $S_P(\mathcal{D}_D^{s,m})$ with $\pi_P^{-1}(U)$, an open subscheme of Drap_D^P, thus proving (2).

Given a polygon P, let P_i, for $0 \leq i \leq l$, be the polygon having as vertices the first $i+1$ vertices (r_0, d_0), ..., (r_i, d_i) of P. Consider the sequence of morphisms

$$S_{P_l}(\mathcal{D}_D^{r_l,d_l}) \xrightarrow{\phi_l} \cdots \rightarrow S_{P_i}(\mathcal{D}_D^{r_i,d_i}) \xrightarrow{\phi_i} S_{P_{i-1}}(\mathcal{D}_D^{r_{i-1},d_{i-1}}) \rightarrow \ldots \longrightarrow S_{P_0}(\mathcal{D}_D^{0,0}) \tag{5.5}$$

defined by

$$\phi_i(E_i) = E_{i-1}$$

whenever

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{i-1} \subset E_i$$

is the Harder-Narasimhan filtration of E_i.

It is easily verified, using the description of the tangent spaces to the strata given above, that the differential of each ϕ_i at E_i

$$(\,\mathrm{d}\phi_i)_{E_i} : T_{E_i} \longrightarrow T_{E_{i-1}}$$

fits into an exact sequence

where $Q(E_i, E_{i-1}) \subset \mathrm{Ext}^1(E_i/E_{i-1}, \widetilde{E}_i)$ is the image of the composite of the obvious maps

$$\mathrm{Hom}\ (E_{i-1}, \widetilde{E}_{i-1}) \longrightarrow \mathrm{Hom}\ (E_{i-1}, \widetilde{E}_i)$$

and

$$\mathrm{Hom}\ (E_{i-1}, \widetilde{E}_i) \longrightarrow \mathrm{Ext}^1(E_i/E_{i-1}, \widetilde{E}_i)\ .$$

The vector space Hom $(E_i/E_{i-1}, \widetilde{E}_i)$ can be identified with the tangent space to the fibre $\phi_i^{-1}(E_{i-1})$ at E_i. Indeed, the fibre $\phi_i^{-1}(F)$ over any F is isomorphic to an open subscheme of the projective scheme parametrizing (r_i', d_i')-submodules of $\widetilde{F}$ (namely, the subscheme consisting of the semistable ones.)

Make now the assumption that $\deg D > \dfrac{d_1'}{r_1'} + 2g - 1$. Since P is convex, this is equivalent to the assumption that $\deg D > \dfrac{d_i'}{r_i'} + 2g - 1$ for $i = 1, \ldots, l$.

Then, as a consequence of (5.1), we have

$$\mathrm{Ext}^1(E_i/E_{i-1}, \widetilde{E}_i) = 0\ .$$

This implies, that the non-empty fibres of ϕ_i are all smooth and have the same dimension. Moreover, the exact sequence above (where $Q(E_i, E_{i-1})$ now vanishes) shows that the differential of ϕ_i is everywhere surjective. It is also not difficult to verify that ϕ_i is surjective (observe that if E_{i-1} is a submodule of $\mathcal{O}_C(D)^r$ having rank r_{i-1}, then there is an injection $\mathcal{O}_C(D)^{r-r_{i-1}} \to \widetilde{E}_{i-1}$ and one can use (5.1) to inject any semistable (r_i', d_i')-bundle into $\mathcal{O}_C(D)^{r_i'}$). It follows, by induction on i, that the strata $S_{P_i}(\mathcal{D}_D^{r_i,s_i})$ are smooth and that each ϕ_i is smooth. The equality codim $S_{P_i}(\mathcal{D}_D^{r_i,s_i}) = \mathrm{d}_{P_i}$ can also be verified by induction on i using the Riemann-Roch theorem and the exact sequence (5.6).

To complete the proof of (3) we still have to verify the assertion made there about the normal bundle. We shall limit ourselves here to showing that the normal space N_E to $S_P(\mathcal{D}_D^{s,m})$ at a point E is naturally isomorphic to $H^1(\mathcal{H}om_+(E, E))$, where $\mathcal{H}om_+(E, E)$ is taken relative to the Harder-Narasimhan filtration of E.

We already know that N_E is isomorphic to $\mathrm{Hom}(E, \widetilde{E})/T_E$, where T_E is the subspace of Hom $(E, \widetilde{E})$ consisting of the homorphisms f such that for $i = 1, \ldots, l$ there are homorphisms $f_i \in$ Hom $(E_i, \widetilde{E}_i)$, with $f_l = f$, making the following diagrams commutative

$$\begin{array}{ccc} E_i & \xrightarrow{f_i} & \widetilde{E}_i \\ \big\downarrow & & \big\downarrow \\ E_{i+1} & \xrightarrow{f_{i+1}} & \widetilde{E}_{i+1} \end{array}$$

for $i = 1, \ldots, l-1$. Consider now the exact sequences

$$H^1(\mathcal{H}om_-(E, E)) \xrightarrow{\alpha} H^1(\mathcal{H}om(E, E)) \longrightarrow H^1(\mathcal{H}om_+(E, E)) \to 0$$

The assumption that $\deg D > \dfrac{d'_i}{r'_i} + 2g - 1$ together with (5.1) imply

$$\mathrm{Ext}^1(E, \widetilde{E}) = 0 \ ,$$

and β is surjective. Therefore we will know that β induces an isomorphism

$$\mathrm{Hom}(E, \widetilde{E})/T_E \xrightarrow{\sim} H^1(\mathcal{H}om_+(E, E))$$

as soon as we show that $\beta^{-1}(\mathrm{Im}\,\alpha) = T_E$. But this is clear once we interpret $\mathrm{Im}\,\alpha$ (resp. T_E) as the space of first order infinitesimal deformations of E (resp. of E inside $\mathcal{O}_C(D)^r$) which, at least in one way, deform the Harder-Narasimhan filtration of E. (In fact there is at most one way to do this: $(f_1, \dots, f_l)$ is, as we know, uniquely determined by f and, similarly, one can show that α is injective.) This completes the proof of (3).

We claim that for every $E \in \mathcal{D}_D^{s,m}$, with Harder-Narasimhan filtration as in (5.4), one has

$$\sum_{i=1}^{l} \dim\, \mathrm{Ext}^1(E_i/E_{i-1}, \widetilde{E}_i) \leq c \tag{5.7}$$

where c is a constant that depends only on r. Observe that

$$\dim \mathrm{Ext}^1(E_i/E_{i-1}, \widetilde{E}_i) \leq r \dim H^1((E_i/E_{i-1})^\vee \otimes \mathcal{O}_C(D)) \ . \tag{5.8}$$

Also

$$\begin{aligned} \mu((E_i/E_{i-1})^\vee \otimes \mathcal{O}_C(D)) = \deg D - \frac{d'_i}{r'_i} \geq \\ \geq \deg D - \frac{d_1}{r_1} \geq 0 \ . \end{aligned}$$

and, if

$$H^1((E_i/E_{i-1})^\vee \otimes \mathcal{O}_C(D)) \neq 0 \tag{5.9}$$

then, because of (5.1), one has

$$\mu((E_i/E_{i-1})^\vee \otimes \mathcal{O}_C(D)) \leq 2g - 1.$$

It follows that the modules $(E_i/E_{i-1})^\vee \otimes \mathcal{O}_C(D)$ satisfying (5.9) vary in a bounded family. This together with (5.8) implies (5.7). Now, taking into account the exact sequence (5.6) and using (5.7) we conclude that

$$\mathrm{codim}\ S_P(\mathcal{D}_D^{s,m}) \geq \ \mathrm{d}_P - c \ . \tag{5.10}$$

Because of the way d_P is defined and because P is convex, we also have

$$\mathrm{d}_P \geq \frac{d'_1}{r'_1} - \frac{d'_l}{r'_l} - 1 \geq \frac{d'_1}{r'_1} - \frac{m}{s} - 1 \ .$$

If $\deg D \leq \dfrac{d'_1}{r'_1} + 2g - 1$, then we obtain

$$\mathrm{d}_P - c \geq \deg D - c - \frac{m}{s} - 2g \ .$$

6. A digression on moduli spaces of rigidified semistable bundles

Although there is not even a coarse moduli space for semistable bundles, there is always a fine moduli space for rigidified semistable bundles. We shall need this result in the following section.

Let $\mathbf{x}$ be a finite nonempty set of points on the curve C. A family, parametrized by a scheme S, of $\mathbf{x}$-rigidified vector bundles of rank r over C is a pair (E, u) where E is a vector bundle of rank r over $C \times S$ and u is an isomorphism $\mathcal{O}_X^r \to E|_X$, $X = \mathbf{x} \times S$.

Proposition 6.1. *There exists a universal family of $\mathbf{x}$-rigidified semistable bundles of rank r and degree d. We shall denote $M(r, d\,; \mathbf{x})$ the parameter space of this family.*

Proof. As is well known [25, 26] there is a family $\mathcal{F}$ of semistable (r, d)-bundles over C which is parametrized by a smooth quasi-projective variety T and has compatible actions of $\mathrm{GL}(N)$ on both $\mathcal{F}$ and T having the following properties:

a) $\mathcal{F}$ has the local universal property for families of semistable bundles of rank r and degree d over C;
b) If $t_1, t_2 \in T$, then $\mathcal{F}_{t_1}$ is isomorphic to $\mathcal{F}_{t_2}$ if, and only if, t_1 and t_2 are in the same orbit;
c) the stabilizer of $t \in T$ maps isomorphically onto the group of automorphisms of $\mathcal{F}_t$;
d) a good quotient of T under the action of $GL(N)$ exists.

For each $x \in \mathbf{x}$, let $q_x : \mathcal{B}(x) \longrightarrow T$ be the fibre bundle whose fibre over $t \in T$ is naturally isomorphic to the variety of all bases of the vector space $\mathcal{F}(x, t)$. Set

$$\mathcal{B}(\mathbf{x}) = \prod_{x \in \mathbf{x}}{}_T\ \mathcal{B}(x)$$

and $\tilde{\mathcal{F}} = (1_C \times q)^* \mathcal{F}$, where $q : \mathcal{B}(\mathbf{x}) \longrightarrow T$ is the natural projection.

Now $\tilde{\mathcal{F}}$ is canonically $\mathbf{x}$-rigidified and the action of $GL(N)$ on $\mathcal{F}$ and T lifts to an action on $\tilde{\mathcal{F}}$ and $\mathcal{B}(\mathbf{x})$. Since the map q is affine and because of property d), a good quotient of $\mathcal{B}(\mathbf{x})$ under the action of $GL(N)$ exists. (This will be $M(r, d\,; \mathbf{x})$.) An automorphism of a semistable bundle E that leaves fixed a basis of $E(x)$, for x a point in C, has to be the identity. It follows from this and from property c) that the stabilizer of any $z \in \mathcal{B}(\mathbf{x})$ is the identity. Therefore $M(r, d\,; \mathbf{x})$ is a geometric quotient.

Recall that if $\mathcal{L}$ and $\mathcal{N}$ are two families of semistable $\mathbf{x}$-rigidified bundles parametrized by the same S, then the set

$$S' = \{ s \in S |\ \mathcal{L}_s \text{ is isomorphic to } \mathcal{N}_s \text{ as } \mathbf{x}\text{-rigidified bundles } \}$$

is closed in S and there is an isomorphism of families of $\mathbf{x}$-rigidified bundles

$$\mathcal{L}|_{C \times S'} \xrightarrow{\sim} \mathcal{N}|_{C \times S'}\ .$$

It follows that the action of $GL(N)$ on $\mathcal{B}(\mathbf{x})$ is free, and hence that $\mathcal{B}(\mathbf{x})$ is a $GL(N)$-principal bundle over $M(r, d\,; \mathbf{x})$. At this point, standard criteria for descent allow us to conclude that $\tilde{\mathcal{F}}$ descends to a vector bundle $\mathcal{E}$ over $C \times M(r, d\,; \mathbf{x})$.

7. Cohomology of the strata

Let P be a Shatz polygon having vertices $(r_0, d_0), \ldots, (r_l, d_l)$. We set, as usual, $r'_i = r_i - r_{i-1}$ and $d'_i = d_i - d_{i-1}$ for $i = 1, \ldots, l$. Consider the closed immersion of ind-varieties

$$\boldsymbol{\delta} : (\mathbf{Div}^{r'_1,d'_1}_{C/k})^{ss} \times \cdots \times (\mathbf{Div}^{r'_l,d'_l}_{C/k})^{ss} \longrightarrow \mathbf{S}_P \ .$$

defined by $\boldsymbol{\delta}((F_1, \ldots, F_l)) = F_l \oplus F_{l-1} \oplus \cdots \oplus F_1$.

The aim of this section is to prove the following

Proposition 7.1. *The cohomology of the stratum $\mathbf{S}_P$ stabilizes. Furthermore, the morphism $\boldsymbol{\delta}$ induces an isomorphism in cohomology, and we have an identity of Poincaré series*

$$P(\mathbf{S}_P; t) = \prod_{i=1}^{l} P((\mathbf{Div}^{r'_i,d'_i}_{C/k})^{ss}; t) \ .$$

Before we start proving this proposition it will be useful to have a preliminary result.

Proposition 7.2. *Let $\mathcal{D}^{s,m}_r(D)$ be the projective scheme parametrizing (s,m)-submodules of $\mathcal{O}_C(D)^r$. Also, let $\mathcal{C}^{s,m}_r(D)$ be its closed subset consisting of the submodules L of $\mathcal{O}_C(D)^r = \mathcal{O}_C(D)^{r-s} \oplus \mathcal{O}_C(D)^s$ which do not project injectively into $\mathcal{O}_C(D)^s$.*

We have

$$\dim \mathcal{D}^{s,m}_r(D) \leq r(s \deg D - m) + a \ , \tag{7.1}$$

and

$$\dim \mathcal{C}^{s,m}_r(D) \leq s(r-1) \deg D + b \tag{7.2}$$

where both a and b are constants independent of D.

Moreover, for $P \neq ss$, define $\mathbf{S}^o_P$ as the open subset of the P-stratum consisting of divisors E having Harder-Narasimhan filtration

$$0 = E_0 \subset \cdots \subset E_{l-1} \subset E_l = E$$

such that E_{l-1} projects injectively into the second factor of the direct sum decomposition $\mathcal{O}_C(D)^r = \mathcal{O}_C(D)^{r-r_{l-1}} \oplus \mathcal{O}_C(D)^{r_{l-1}}$.

Then the open immersion

$$\mathbf{S}^o_P \to \mathbf{S}_P$$

is a quasi isomorphism (in particular it induces an isomorphism in cohomology.)

Proof. The Zariski tangent space to the variety $\mathcal{D}^{s,n}_r(D)$ at a point L is isomorphic to Hom $(L, \mathcal{O}_C(D)^r/L)$. We can bound the dimension of this vector space from above using the Riemann-Roch theorem and the obvious inequalities

$$\begin{aligned} \dim \operatorname{Ext}^1(L, \mathcal{O}_C(D)^r/L) &\leq \dim \operatorname{Ext}^1(L, \mathcal{O}_C(D)^r) \\ &\leq \dim \operatorname{Ext}^1(\mathcal{O}_C(D)^r, \mathcal{O}_C(D)^r) \leq r^2 g \ . \end{aligned}$$

The resulting bound gives the first inequality.

Consider now the stratification of $Q = \mathcal{D}^{s,m}_r(D)$ associated to the direct sum

The stratum $Q^{(s_1,m_1;s_2,m_2)}$, where $s_1+s_2=s$ and $m_1+m_2=m$, maps to $\mathcal{D}^{s_1,m_1}_{r-s}(D)\times\mathcal{D}^{s_2,m_2}_{s}(D)$ and the fibre over (L_1,L_2) is isomorphic to $\mathrm{Hom}(L_1,\mathcal{O}_C(D)^{r-s}/L_2)$. $\blacksquare$

We can bound the dimension of $\mathcal{D}^{s_1,m_1}_{r-s}(D)\times\mathcal{D}^{s_2,m_2}_{s}(D)$ using (7.1), and that of the vector space $\mathrm{Hom}(L_1,\mathcal{O}_C(D)^{r-s}/L_2)$ using both the Riemann-Roch theorem and the obvious inequalities

$$\begin{aligned}\dim\mathrm{Ext}^1(L_2,\mathcal{O}_C(D)^{r-s}/L_1)\le\dim\mathrm{Ext}^1(L_2,\mathcal{O}_C(D)^{r-s})\le\\ \le\dim\mathrm{Ext}^1(\mathcal{O}_C(D)^{s},\mathcal{O}_C(D)^{r-s})\le s(r-s)g\ .\end{aligned}$$

As a result, we obtain

$$\dim Q^{(s_1,m_1\,;\,s_2,m_2)}\le(r-s+s_2)s\deg D-(r-s+s_2)m+b\ ,$$

where b is a constant that depends only on r. Since $\mathcal{C}^{s,n}_r(D)$ is, by definition, the union of the strata for which $s_2\le s-1$, the second inequality follows.

Finally, recall the map

$$\phi_l: S_P(D)\longrightarrow S_{P_{l-1}}(\mathcal{D}^{r_{l-1},d_{l-1}}_r(D))$$

used in the proof of Proposition 5.1 If $E\in S_P(D)-S^o_P(D)$, then by definition

$$\phi_l(E)\in S_{P_{l-1}}(\mathcal{D}^{r_{l-1},d_{l-1}}_r(D))\cap\mathcal{C}^{r_{l-1},d_{l-1}}_r(D)\ .$$

We know that, for $\deg D$ large enough, ϕ_l is a smooth surjective map between smooth varieties, and we have

$$\dim S_{P_{l-1}}(\mathcal{D}^{r_{l-1},d_{l-1}}_r(D))=r_{l-1}r\deg D+c$$

with c a constant independent of D. The result now follows from (7.2).

Proof of Proposition 7.1. We shall first prove that $\boldsymbol{\delta}$ induces an isomorphism in cohomology. Let $\widetilde{\mathbf{S}}_P=(\mathbf{Div}^{r_{l-1},d_{l-1}}_{C/k})^{P_{l-1}}$. By induction on l it suffices to prove the result for the map

$$\boldsymbol{\delta}:\widetilde{\mathbf{S}}_P\times(\mathbf{Div}^{r'_l,d'_l}_{C/k})^{ss}\longrightarrow\mathbf{S}_P \tag{7.3}$$

defined by $\boldsymbol{\delta}((F,G))=G\oplus F$. Since the image of $\boldsymbol{\delta}$ is contained in $\mathbf{S}^o_P$, we can substitute $\mathbf{S}^o_P$ for $\mathbf{S}_P$ in (7.1).

We shall show that for every $i\ge 0$, there is an integer N such that given any D, with $\deg D>N$, one can find open sets A and V, of the elements of index D in the left and right hand sides of (7.3), so that in the commutative diagram

$$\begin{tikzcd} H^i(S_P(D)) \arrow[r, "\boldsymbol{\delta}^*"] \arrow[d] & H^i(\widetilde{S}_P(D)\times(\mathrm{Div}^{r'_l,d'_l}_{C/k}(D))^{ss}) \arrow[d] \\ H^i(V) \arrow[r, "\boldsymbol{\delta}^*"] & H^i(A) \end{tikzcd}$$

Let D be an effective divisor on C and let $\mathbf{x}$ be a nonempty set of points in C not lying on the support of D. Fix the direct sum decomposition

$$\mathcal{O}_C(D)^r = \mathcal{O}_C(D)^{r'_l} \oplus \mathcal{O}_C(D)^{r_{l-1}}$$

and define the following open sets:

(1) $A^{(1)}_{\mathbf{x}}(D) \subset \widetilde{S}_P(D)$ consists of the divisors F such that the support of $\mathcal{O}_C(D)^{r_{l-1}}/F$ is disjoint from $\mathbf{x}$.
(2) $A^{(2)}_{\mathbf{x}}(D) \subset (\mathrm{Div}^{r'_l,d'_l}_{C/k}(D))^{ss}$ consists of the divisors G such that the support of $\mathcal{O}_C(D)^{r'_l}/G$ does not intersect $\mathbf{x}$.
(3) $V_{\mathbf{x}}(D) \subset S^o_P(D)$ consists of the divisors E, having Harder-Narasimhan filtration

$$0 = E_0 \subset \cdots \subset E_{l-1} \subset E_l = E \ , \tag{7.4}$$

such that the points of $\mathbf{x}$ do not lie on the supports of the quotients $\mathcal{O}_C(D)^r/E$ and $\mathcal{O}_C(D)^{r_{l-1}}/\overline{E}_{l-1}$. Here $\overline{E}_{l-1}$ is the image of E_{l-1} under the projection of $\mathcal{O}_C(D)^r$ into $\mathcal{O}_C(D)^{r_{l-1}}$.
(4) $W_{\mathbf{x}}(D) \subset (\mathcal{D}^{r_{l-1},d_{l-1}}_r(D))^{P_{l-1}}$ consists of the submodules L of $\mathcal{O}_C(D)^r$ such that L projects injectively onto a submodule $\overline{L}$ of $\mathcal{O}_C(D)^{r_{l-1}}$ and the support of $\mathcal{O}_C(D)^{r_{l-1}}/\overline{L}$ does not intersect $\mathbf{x}$.

Proposition 7.3. *Let*

$$N_q = \max \{ \frac{d'_1}{r'_1} + 2g - 1, \ \frac{d'_l}{r'_l} + 2g - 1 + q\} \tag{7.5}$$

for $q \geq 0$. Let $\mathbf{x}$ be a nonempty set of points in C and set $N = N_{\#\mathbf{x}}$, where $\#\mathbf{x}$ is the cardinality of $\mathbf{x}$. If D is an effective divisor with $\deg D > N$, then the open sets just defined have the following properties:

(1) *If we set $A_{\mathbf{x}}(D) = A^{(1)}_{\mathbf{x}}(D) \times A^{(2)}_{\mathbf{x}}(D)$, then*

$$\boldsymbol{\delta} : A_{\mathbf{x}}(D) \longrightarrow V_{\mathbf{x}}(D)$$

induces an isomorphism of cohomology groups.
(2) *If we set*

$$A^{\mathbf{x}}_D = \bigcup_{x \in \mathbf{x}} A_{\{x\}}(D)$$

and

$$V^{\mathbf{x}}_D = \bigcup_{x \in \mathbf{x}} V_{\{x\}}(D) \ ,$$

then

$$\boldsymbol{\delta} : A^{\mathbf{x}}_D \longrightarrow V^{\mathbf{x}}_D$$

also induces an isomorphism of cohomology groups.
(3) *The codimensions of the complements of both $A^{\mathbf{x}}_D$ and $V^{\mathbf{x}}_D$ are not less than the integral part of half the cardinality of $\mathbf{x}$. In particular, for i smaller*

It follows that the open sets A and V, mentioned before, may be obtained by choosing a set $\mathbf{x}$ of more than i distinct points in C, not appearing on D, and setting $A = A_D^{\mathbf{x}}$, $V = V_D^{\mathbf{x}}$.

Proof. (1) Let $M = M(r_l', d_l'; \mathbf{x})$ be the moduli space of $\mathbf{x}$-rigidified semistable bundles, having rank r_l' and degree d_l', that was considered in the previous section.

Our strategy will be to construct a commutative diagram

$$\begin{array}{ccc} A_{\mathbf{x}}^{(1)}(D) \times A_{\mathbf{x}}^{(2)}(D) & \xrightarrow{\delta} & V_{\mathbf{x}}(D) \\ {\scriptstyle 1\times\varphi}\downarrow & & \downarrow{\scriptstyle (\pi,\psi)} \\ A_{\mathbf{x}}^{(1)}(D) \times M & \xrightarrow[j\times 1]{} & W_{\mathbf{x}}(D) \times M \end{array}$$

such that the top varieties are affine bundles over the bottom ones and j induces an isomorphism in cohomology. (See, for example, Proposition 5.12 and Theorem 5.15 in Chapter VI of [23] for the acyclicity of affine bundles.)

Let us now describe the maps that appear in the diagram above.

Any G in $A_{\mathbf{x}}^{(2)}(D)$ is naturally $\mathbf{x}$-rigidified as follows. Consider the short exact sequences

$$0 \to \mathcal{O}_C^{r_l'} \to \mathcal{O}_C(D)^{r_l'} \to \mathcal{O}_D^{r_l'} \to 0 \tag{7.6}$$

and

$$0 \to G \to \mathcal{O}_C(D)^{r_l'} \to \mathcal{O}_C(D)^{r_l'}/G \to 0 \ .$$

It is clear that their restriction to $\mathbf{x}$ provide us with an $\mathbf{x}$-rigidification u determined by the isomorphisms

$$\mathcal{O}_{\mathbf{x}}^{r_l'} \to \mathcal{O}_C(D)^{r_l'}|_{\mathbf{x}} \leftarrow G|_{\mathbf{x}} \ .$$

The map $\varphi : A_{\mathbf{x}}^{(2)}(D) \to M$ sends G to $[G, u]$, the isomorphism class of (G, u).

Similarly, the map $\psi : V_{\mathbf{x}}(D) \to M$ associates to any E in $V_{\mathbf{x}}(D)$ the isomorphism class of $(E/E_{l-1}, u)$ where E_{l-1} is as in (7.2) and u is the $\mathbf{x}$-rigidification obtained as follows. The short exact sequences

$$0 \to \mathcal{O}_C(D)^{r_l'} \to \mathcal{O}_C(D)^r/E_{l-1} \to \mathcal{O}_C(D)^{r_{l-1}}/\overline{E}_{l-1} \to 0$$

and

$$0 \to E/E_{l-1} \to \mathcal{O}_C(D)^r/E_{l-1} \to \mathcal{O}_C(D)^r/E \to 0$$

along with (7.6) provide us, after restriction to $\mathbf{x}$, with the chain of isomorphisms

$$\mathcal{O}_{\mathbf{x}}^{r_l'} \to \mathcal{O}_C(D)^{r_l'}|_{\mathbf{x}} \to (\mathcal{O}_C(D)^r/E_{l-1})|_{\mathbf{x}} \leftarrow (E/E_{l-1})|_{\mathbf{x}}$$

which defines u.

The map $\pi : V_{\mathbf{x}}(D) \to W_{\mathbf{x}}(D)$ associates to a divisor E the term E_{l-1} of its Harder-Narasimhan filtration (7.4).

Finally, the map $j : A_{\mathbf{x}}^{(2)}(D) \to W_{\mathbf{x}}(D)$ is defined by $j(M) = 0 \oplus M$ where $0 \oplus M \subset \mathcal{O}_C(D)^{r_l'} \oplus \mathcal{O}_C(D)^{r_{l-1}}$.

canonically isomorphic to Hom $(F, \mathcal{O}_C(D)^r/L)$. (Note that $\deg D > N$ implies $\mathrm{Ext}^1(F, \mathcal{O}_C(D)^r/L) = 0$.)

One can lift (π, ψ) to a map

$$\lambda : V_{\mathbf{x}}(D) \to \mathcal{H}_{\mathbf{x}}(D)$$

which assigns to E the homomorphism $E/E_{l-1} \to \mathcal{O}_C(D)^r/E_{l-1}$ deduced from the inclusion $E \subseteq \mathcal{O}_C(D)^r$. The map λ is clearly injective.

We claim that λ defines an isomorphism of $V_{\mathbf{x}}(D)$ with an affine subbundle of $\mathcal{H}_{\mathbf{x}}(D)$.

Let $(L, [F, u])$ be a point of $W_{\mathbf{x}}(D) \times M$ and let $\mu : F|_{\mathbf{x}} \to G|_{\mathbf{x}}$, $G = \mathcal{O}_C(D)^r/L$, be the composition of the chain of isomorphisms

$$F|_{\mathbf{x}} \xrightarrow{u^{-1}} \mathcal{O}_{\mathbf{x}}^{r'_l} \to \mathcal{O}_C(D)^{r'_l}|_{\mathbf{x}} \to G|_{\mathbf{x}}$$

Let i be the inclusion of $\mathbf{x}$ in C and let

$$\mu_0 : F \longrightarrow i_* i^* G \simeq G \otimes i_* \mathcal{O}_{\mathbf{x}}$$

be the composition of $F \to i_* i^* F$ and $i_* \mu$.

It is easily verified that the elements of Hom (F, G) in the image of λ are precisely those which map to μ_0 under the map

$$\mathrm{Hom}\ (F, G) \longrightarrow \mathrm{Hom}\ (F, G \otimes i_* \mathcal{O}_{\mathbf{x}})\ .$$

Since $\deg D > N$ we have

$$\mathrm{Ext}^1(F, G \otimes \mathcal{O}_C(-\mathbf{x})\,) = 0$$

and the sequence

$$0 \to \mathrm{Hom}\ (F, G \otimes \mathcal{O}_C(-\mathbf{x})) \to \mathrm{Hom}\ (F, G) \to \mathrm{Hom}\ (F, G \otimes i_* \mathcal{O}_{\mathbf{x}}) \to 0$$

is exact.

Thus the image of λ intersects every fibre of the vector bundle $\mathcal{H}_{\mathbf{x}}(D)$ in an affine space of constant dimension and is therefore an affine bundle over $W_{\mathbf{x}}(D) \times M$. It is not difficult to verify that the differential of λ is everywhere injective (note that the tangent space to M at $[F, u]$ can be identified with $\mathrm{Ext}^1(F, F(-\mathbf{x}))$.) It follows that λ is a closed immersion and the claim is verified.

In a similar way one can prove that $\varphi : A^{(2)}_{\mathbf{x}}(D) \longrightarrow M$ is also an affine bundle.

We now turn to the proof that $j : A^{(1)}_{\mathbf{x}}(D) \longrightarrow W_{\mathbf{x}}(D)$ induces an isomorphism in cohomology. Consider the retraction $r : W_{\mathbf{x}}(D) \longrightarrow A^{(1)}_{\mathbf{x}}(D)$ defined by $r(L) = \overline{L}$, where $\overline{L}$ is the projection of $L \subseteq \mathcal{O}_C(D)^r$ into $\mathcal{O}_C(D)^{r_{l-1}}$. The fibre of r over $\overline{L}$ is isomorphic to Hom $(\overline{L},\, \mathcal{O}_C(D)^{r'_l})$. Since $\deg D > N$ we have $\mathrm{Ext}^1(\overline{L},\, \mathcal{O}_C(D)^{r'_l}) = 0$, and $W_{\mathbf{x}}(D)$ is a vector bundle over $A^{(1)}_{\mathbf{x}}(D)$ having j as its zero section. This proves what we wanted.

(2) Observe that if $\mathbf{y}$ is another nonempty set of distinct points of C, then

and

$$V_{\mathbf{x}}(D) \cap V_{\mathbf{y}}(D) = V_{\mathbf{x}\cup\mathbf{y}}(D)\ .$$

Comparing obvious Mayer-Vietoris sequences and using the five-lemma, we conclude that $\boldsymbol{\delta} : A_D^{\mathbf{x}} \longrightarrow V_D^{\mathbf{x}}$ gives an isomorphism in cohomology.

(3) We shall only consider the complement of $V_D^{\mathbf{x}}$, but the same argument applies to that of $A_D^{\mathbf{x}}$. Remark that if $E \in S_P^o(D) - V_D^{\mathbf{x}}$, then at least half the points of $\mathbf{x}$ belong simultaneously either to the support of $\mathcal{O}_C(D)^r/E$ or to that of $\mathcal{O}_C(D)^{r_{l-1}}/\overline{E}_{l-1}$. We are thus reduced to proving the following assertion.

Let $\mathbf{y}$ be a nonempty set of points in C. If $Z_{\mathbf{y}}(D)$ is the subset of $\mathrm{Div}^{r,n}_{C/k}(D)$ consisting of the divisors E such that $\mathbf{y}$ is contained in the support of $\mathcal{O}_C(D)^r/E$, then the codimension of $Z_{\mathbf{y}}(D)$ is greater, or equal, than the number of elements in $\mathbf{y}$.

This can be established by induction on r using the stratification of $\mathrm{Div}^{r,n}_{C/k}(D)$ associated to the direct sum decomposition $\mathcal{O}_C(D)^r = \mathcal{O}_C(D) \oplus \mathcal{O}_C(D)^{r-1}$.

To complete the proof of Proposition 7.1 we still have to show that the cohomology of $\mathbf{S}_P$ stabilizes.

Since $\boldsymbol{\delta}$ induces an isomorphism in cohomology, it suffices to show that the cohomology of $(\mathbf{Div}^{r,n}_{C/k})^{ss}$ stabilizes.

Given $i \geq 0$, let $N = N_{i+1}$ be as in (7.5). If D, D' are effective divisors, with $D' \geq D$ and $\deg D > N$, then choose $\mathbf{x}$ to consist of $i+1$ distinct points in C not lying on D'. Now, in the commutative diagram

$$\begin{array}{ccc}
(\mathbf{Div}^{r,n}_{C/k}(D))^{ss} & \to & (\mathbf{Div}^{r,n}_{C/k}(D'))^{ss} \\
\uparrow & & \uparrow \\
A^{(2)}_{\mathbf{x}}(D) & \to & A^{(2)}_{\mathbf{x}}(D') \\
\varphi \searrow & & \swarrow \varphi' \\
 & M &
\end{array}$$

both φ and φ' are affine bundles over M. It follows that the top arrow induces an isomorphism of i-th cohomology groups, and we are done.

8. The Abel-Jacobi map

Recall that there is a coarse moduli space $N(r,n)$ parametrizing isomorphism classes of stable vector bundles of rank r and degree n over the curve C. This is a smooth quasi-projective variety of dimension $1 + r^2(g-1)$. When r and n are coprime the notion of stable and semistable bundle coincide and $N(r,n)$ is a smooth projective algebraic variety. Moreover, in this case, $N(r,n)$ is a fine moduli space. In particular there are Poincaré bundles $\mathfrak{P}^{r,n}$ over $C \times N(r,n)$ such that for every $[E] \in N(r,n)$ the restriction $\mathfrak{P}^{r,n}_{[E]}$ of $\mathfrak{P}^{r,n}$ to $C \times \{[E]\}$ is isomorphic to E.

Let r and n be coprime. It is natural to define, by analogy with the classical case, Abel-Jacobi maps

$$\boldsymbol{\vartheta} : (\mathbf{Div}^{r,n}_{C/k})^{ss} \longrightarrow N(r,n)\ ,$$

by assigning to a divisor E its isomorphism class as a vector bundle. Here $N(r,n)$

Proposition 8.1. *If r and n are coprime, then there is a "coherent locally free" module $\boldsymbol{E}$, defined over the constant ind-variety $N(r,n)$, and a morphism*

$$\begin{array}{ccc} (\mathbf{Div}^{r,n}_{C/k})^{ss} & \xrightarrow{\mathbf{i}} & \mathbb{P}(\boldsymbol{E}) \\ \boldsymbol{\vartheta} \quad \searrow & & \swarrow \\ & N(r,n) & \end{array}$$

inducing an isomorphism in cohomology. In particular,

$$H^*((\mathbf{Div}^{r,n}_{C/k})^{ss}) = H^*(N(r,n))[x] \ ,$$

where x is an independent variable of degree 2. This gives an identity of Poincaré series

$$P(N(r,n);t) = (1-t^2)P((\mathbf{Div}^{r,n}_{C/k})^{ss};t) \ .$$

Proof. For simplicity, we shall work over the cofinal subset of positive divisors such that

$$\deg D > 2g + \frac{n}{r} - 1 \ .$$

We construct, using the Poincaré bundle, a vector bundle $\mathcal{H}(D)$ over $N(r,n)$ whose fibre at any $[E]$ is canonically isomorphic to Hom $(\mathfrak{P}_{[E]}, \mathcal{O}_C(D)^r)$. (Recall that for any stable (r,n)-bundle F, one has $\mathrm{Ext}^1(F, \mathcal{O}_C(D)^r) = 0$.)

There is a natural morphism $i_D : (\mathrm{Div}^{r,n}_{C/k}(D)))^{ss} \longrightarrow \mathbb{P}(\mathcal{H}(D))$ defined by sending any E to the class in $\mathbb{P}(\mathrm{Hom}\ (\mathfrak{P}_{[E]}, \mathcal{O}_C(D)^r))$ of the composite of any isomorphism $\mathfrak{P}_{[E]} \longrightarrow E$ with the inclusion $E \to \mathcal{O}_C(D)^r$. Recall that the automorphisms of a stable bundle are given by multiplication by non-zero scalars.

(We leave to the conscientious reader the task of defining $\mathcal{H}(D)$, $\mathbf{i}$ using E.G.A. III §7.7 and the universal properties of the different objects under consideration.)

The map i_D is injective by definition. Also, it is not difficult to verify that i_D is an immersion (let us mention here that the differential of i_D at E can be naturally identified with the coboundary map Hom $(E, \mathcal{O}_C(D)^r/E) \longrightarrow \mathrm{Ext}^1(E,E)$). Furthermore it is clear that the image of i_D intersects any fibre $\mathbb{P}(\mathrm{Hom}\ (\mathfrak{P}_{[E]}, \mathcal{O}_C(D)^r))$ in the open subset consisting of the classes of injective homomorphisms.

We claim that $\mathbf{i}$ is a quasi isomorphism. This follows from the following

Lemma 8.2. *Let E, F be locally free $\mathcal{O}_C$-modules of rank r such that* $\mathrm{Ext}^1(E,F)$ *vanishes. For any effective divisor D let c_D be the codimension in* Hom $(E, F(D))$ *of the closed subset consisting of the homomorphisms which are not injective. Then we have*

$$c_D \geq \deg D \ .$$

Proof. Since $F \otimes \mathcal{O}_D$ is isomorphic to $\mathcal{O}_D^r$, we have an exact sequence

$$0 \longrightarrow F \longrightarrow F(D) \longrightarrow \mathcal{O}_D^r \longrightarrow 0 \ .$$

From this we obtain another exact sequence

If $\lambda \in \mathrm{Hom}\ (E, F(D))$ is not injective then the rank of the image of the homomorphism $\Phi(\lambda)$ is smaller than r, and $\Phi(\lambda)$ factors through a surjection $E \longrightarrow \mathcal{O}_D^s$ for some $s < r$. Thus, in order to establish the lemma, it suffices to show that the subset of homomorphisms having this propriety has codimension not less than $\deg D$ in $\mathrm{Hom}\ (E, \mathcal{O}_D^r)$.

Note that if $D = \sum_{i=1}^q n_i P_i$, then $\mathcal{O}_D^r \cong \oplus_{i=1}^q \mathcal{O}_{D_i}^r$, with $D_i = n_i P_i$, and

$$\mathrm{Hom}\ (E, \mathcal{O}_D^r) \cong \oplus_{i=1}^q \mathrm{Hom}\ (E, \mathcal{O}_{D_i}^r)\ .$$

Hence we may assume that $D = m\, P$. Let Ψ^s be the map

$$\mathrm{Hom}^{\mathrm{Surj}}(E, \mathcal{O}_D^s) \times \mathrm{Hom}\ (\mathcal{O}_D^s, \mathcal{O}_D^r) \longrightarrow \mathrm{Hom}\ (E, \mathcal{O}_D^r)$$

given by composition. The group G of automorphisms of $\mathcal{O}_D^s$, where $D = mP$, has dimension $s^2 m$ and acts freely on the domain of Ψ^s by $g(\alpha, \beta) = (g \circ \alpha, \beta \circ g^{-1})$. Since Ψ^s is clearly constant along the orbits, we obtain

$$\dim \mathrm{Im}\ \Psi^s \leq 2rsm - s^2 m$$

and

$$\mathrm{codim}\ \mathrm{Im}\, \Psi^s \geq (r-s)^2 m$$

which proves the lemma.

9. The fundamental action on the ind-variety of divisors

Let G be the group of automorphisms of the constant $\mathcal{O}_C$-module K^r. Clearly G is isomorphic to $\mathrm{GL}(r, K)$. We would like to construct an ind-variety $\mathbf{G}$ having G as its k-rational points. We proceed as follows.

First, for any positive divisor D, we set

$$G(D) = \{g \in G \mid g(\mathcal{O}_C^r) \subseteq \mathcal{O}_C(D)^r\}\ .$$

Clearly

$$G = \bigcup_{D \geq 0} G(D)\ .$$

Note that the restriction function

$$G(D) \longrightarrow \mathrm{Hom}^{\mathrm{Inj}}(\mathcal{O}_C^r, \mathcal{O}_C(D)^r)$$

is a bijection (an automorphism of K^r is determined by its restriction to $\mathcal{O}_C^r$ and, conversely, any injective homomorphism $\mathcal{O}_C^r \longrightarrow K^r$ extends to an automorphism of K^r.) It follows that we can identify $G(D)$ with the k-points of an open subvariety of the affine space $\mathbb{V}(E_D)$ determined by the k-vector space $E_D = \mathrm{Hom}\ (\mathcal{O}_C^r, \mathcal{O}_C(D)^r)$.

If $D \leq D'$, then the inclusion $G(D) \hookrightarrow G(D')$ is induced by a closed immersion of the corresponding varieties. The resulting inductive system is $\mathbf{G}$.

The composition law $G \times G \to G$ is induced from a morphism of algebraic ind-

determined by the obvious morphisms of algebraic varieties

$$G(D) \times G(D') \longrightarrow G(D + D') \ ,$$

one for each pair (D, D') of positive divisors.

This makes $\mathbf{G}$ into an ind-variety in monoids (note, however, that if $i : G \to G$ is the map giving the inverse, then for any D there is no D' such that $i(G(D)) \subseteq G(D')$.)

We also define an ind-variety in monoids $\overline{\mathbf{G}}$ corresponding to $\overline{G} = G/k^*$. One has

$$\overline{G} = \bigcup_{D \geq 0} \overline{G}(D)$$

where $\overline{G}(D) = G(D)/k^*$. Since $\overline{G}(D)$ can be identified with the k-points of an open subvariety of $\mathbb{P}(E_D)$, namely the image of $E_D^{\mathrm{Inj}} = \mathrm{Hom}^{\mathrm{Inj}}(\mathcal{O}_C^r, \mathcal{O}_C(D)^r)$ under the projection

$$\mathbb{V}(E_D) - \{0\} \longrightarrow \mathbb{P}(E_D) \ ,$$

we can take $\overline{\mathbf{G}}$ to be the system determined by these subvarieties. Note that if $\mathbf{E} = \{E_D\}_{D \geq 0}$ is the obvious module over the constant ind-variety Spec k, then $\mathbf{G}$ (resp. $\overline{\mathbf{G}}$) is an open ind-subvariety of $\mathbf{V} = \mathbb{V}(\mathbf{E})$ (resp. $\mathbf{P} = \mathbb{P}(\mathbf{E})$.)

Proposition 9.1. *Let $\boldsymbol{\tau}$ be the restriction of $\mathcal{O}_{\mathbf{P}}(1)$ to the open ind-subvariety $\overline{\mathbf{G}}$. The cohomologies of $\mathbf{G}$ and $\overline{\mathbf{G}}$ stabilize. Moreover, if t is the Chern class of $\boldsymbol{\tau}$, then*

$$H^*(\mathbf{G}) = \mathbb{Z}_\ell$$

and

$$H^*(\overline{\mathbf{G}}) = \mathbb{Z}_\ell\,[\,t\,] \ ,$$

with t algebraically independent over $\mathbb{Z}_\ell$.

Proof. Both the inclusions $\mathbf{G} \hookrightarrow \mathbf{V}$ and $\overline{\mathbf{G}} \hookrightarrow \mathbf{P}$ are quasi-isomorphisms (proof as in Lemma 8.2.)

There is a natural action of G on the set of (r,n)-divisors $\mathrm{Div}_{C/k}^{r,n}$ given by $g \cdot E = g(E)$. The orbits of $\mathrm{Div}_{C/k}^{r,n}$ under this action are in one-to-one correspondence with the set of isomorphism classes of (r,n)-vector bundles over C. Note that any such bundle can be injected into K^r and that any isomorphism between two (r,n)-divisors can be extended to an automorphism of K^r.

The action $G \times \mathrm{Div}_{C/k}^{r,n} \longrightarrow \mathrm{Div}_{C/k}^{r,n}$ is induced from an action

$$\boldsymbol{\alpha} : \mathbf{G} \times \mathbf{Div}_{C/k}^{r,n} \longrightarrow \mathbf{Div}_{C/k}^{r,n}$$

determined by the obvious morphisms of algebraic varieties

$$\alpha_{D,D'} : G(D) \times \mathrm{Div}_{C/k}^{r,n}(D') \longrightarrow \mathrm{Div}_{C/k}^{r,n}(D + D') \ ,$$

one for each pair (D, D') of positive divisors. Since k^* acts trivially on $\mathbf{Div}_{C/k}^{r,n}$, we

Let $\boldsymbol{\mathcal{U}}^{r,n}$ be the universal (r,n)-divisor over $C \times \mathbf{Div}^{r,n}_{C/k}$. This is obtained, as usual, from the universal quotients over the components of $\mathbf{Div}^{r,n}_{C/k}$. We shall also denote $\boldsymbol{\mathcal{U}}^{r,n}$ the associated bundle $\mathbb{V}(\boldsymbol{\mathcal{U}}^{r,n})$. This bundle has an obvious $\mathbf{G}$-linearization

$$\tilde{\boldsymbol{\alpha}} : \mathbf{G} \times \boldsymbol{\mathcal{U}}^{r,n} \longrightarrow \boldsymbol{\mathcal{U}}^{r,n}$$

which is compatible with $\boldsymbol{\alpha}$.

Let $\mathbf{S}$ be an ind-variety having an action

$$\boldsymbol{\beta} : \overline{\mathbf{G}} \times \mathbf{S} \longrightarrow \mathbf{S}$$

of $\overline{\mathbf{G}}$ on it (e.g. $\mathbf{S} = \mathbf{S}_P$ with the action induced by $\overline{\boldsymbol{\alpha}}$.)

If $\mathbf{f} : \mathbf{S} \longrightarrow \mathbf{Div}^{r,n}_{C/k}$ is a $\overline{\mathbf{G}}$-equivariant morphism, then we set

$$\boldsymbol{\mathcal{V}} = (1_C \times \mathbf{f})^* \boldsymbol{\mathcal{U}}^{r,n} \ .$$

Given points x_0 in C and s_0 in $\mathbf{S}$ respectively, we consider the sections

$$C \xrightarrow{\boldsymbol{\eta}_{s_0}} C \times \mathbf{S} \xleftarrow{\boldsymbol{\epsilon}_{x_0}} \mathbf{S}$$

defined by $\boldsymbol{\eta}_{s_0}(x) = (x, s_0)$ and $\boldsymbol{\epsilon}_{x_0}(s) = (x_0, s)$.

Also let

$$\overline{\mathbf{G}} \xleftarrow{\mathbf{q}_1} C \times \overline{\mathbf{G}} \times \mathbf{S} \xrightarrow{\mathbf{q}_2} C \times \mathbf{S} \quad ,$$

$$\overline{\mathbf{G}} \xleftarrow{\mathbf{p}_1} \overline{\mathbf{G}} \times \mathbf{S} \xrightarrow{\mathbf{p}_2} \mathbf{S}$$

and

$$C \xleftarrow{\mathbf{u}_1} C \times \overline{\mathbf{G}} \xrightarrow{\mathbf{u}_2} \overline{\mathbf{G}}$$

be the natural projections.

Lemma 9.2. *With the above notations, we have:*

(1) $(1_C \times \boldsymbol{\beta})^* \boldsymbol{\mathcal{V}} \ = \ \mathbf{q}_1^* \boldsymbol{\tau} \otimes \mathbf{q}_2^* \boldsymbol{\mathcal{V}}$
(2) $\boldsymbol{\beta}^* \boldsymbol{\epsilon}_{x_0}^* \boldsymbol{\mathcal{V}} \ = \ \mathbf{p}_1^* \boldsymbol{\tau} \otimes \mathbf{p}_2^* \boldsymbol{\epsilon}_{x_0}^* \boldsymbol{\mathcal{V}}$
(3) $(1_C \times (\boldsymbol{\beta} \circ \boldsymbol{\eta}_{s_0}))^* \boldsymbol{\mathcal{V}} \ = \ \mathbf{u}_2^* \boldsymbol{\tau} \otimes \mathbf{u}_1^* \boldsymbol{\eta}_{s_0}^* \boldsymbol{\mathcal{V}}$.

Proof. (2) and (3) are immediate consequences of (1). By functoriality, it suffices to consider the case where $\mathbf{S} = \mathbf{Div}^{r,n}_{C/k}$ and $\mathbf{f}$ is the identity morphism. For $D \leq D'$, there is a commutative diagram

$$\begin{array}{ccc} G(D) \times \mathcal{U}^{r,n}(D') & \xrightarrow{\tilde{\alpha}_{D,D'}} & \mathcal{U}^{r,n}(D+D') \\ \downarrow & & \downarrow \\ C \times \overline{G}(D) \times \mathrm{Div}^{r,n}_{C/k}(D') & \xrightarrow[1_C \times \overline{\alpha}_{D,D'}]{} & C \times \mathrm{Div}^{r,n}_{C/k}(D+D') \ . \end{array}$$

and we can identify $G(D) \times \mathcal{U}^{r,n}(D')$ with the complement of the zero section in the bundle $(q_1)^*_{D,D'}\tau(D) \otimes (q_2)^*_{D,D'}\mathcal{U}^{r,n}(D')$. Moreover $\tilde{\alpha}_{D,D'}$ extends across the zero section to give an isomorphism

$$(q_1)^*_{D,D'}\tau(D) \otimes (q_2)^*_{D,D'}\mathcal{U}^{r,n}(D') \longrightarrow (1_C \times \overline{\alpha}_{D,D'})^*\mathcal{U}^{r,n}(D+D')$$

Proposition 9.3. *In the situation of the previous lemma let ℓ be prime with r (or, if $k=\mathbb{C}$ and singular cohomology is being used, assume that the coefficient field is of characteristic prime with r.) Then there exists an element x in $H^2(\mathbf{S})$ such that*

$$\boldsymbol{\beta}^*(x) = \mathbf{p}_1^*(t) + \mathbf{p}_2^*(x) \ .$$

Proof. As in the previous proof we can assume that $\mathbf{S} = \mathbf{Div}^{r,n}_{C/k}$. Let y be the first Chern class of $\boldsymbol{\epsilon}^*_{x_0}\boldsymbol{\mathcal{U}}^{r,n}$. Then using 2), in the previous lemma, we have

$$\overline{\boldsymbol{\alpha}}^*(y) = r\cdot \mathbf{p}_1^*(t) + \mathbf{p}_2^*(y) \ .$$

Now take $x = \frac{1}{r}\cdot y$, and we are done.

Definition 9.4. Let $\boldsymbol{\beta} : \overline{\mathbf{G}}\times\mathbf{S} \longrightarrow \mathbf{S}$ be an action of $\overline{\mathbf{G}}$ on the ind-variety $\mathbf{S}$. The ring of invariants in $H^*(\mathbf{S})$ is defined by

$$H^*(\mathbf{S})^{\overline{\mathbf{G}}} = \{\xi\in H^*(\mathbf{S}) \mid \boldsymbol{\beta}^*(\xi) = \mathbf{p}_2^*(\xi)\} \ .$$

Proposition 9.5. *In the situation of the previous definition let $x\in H^2(\mathbf{S})$ be such that*

$$\boldsymbol{\beta}^*(x) = \mathbf{p}_1^*(t) + \mathbf{p}_2^*(x) \ .$$

Then we have

$$H^*(\mathbf{S}) = H^*(\mathbf{S})^{\overline{\mathbf{G}}}[\,x\,] \tag{9.1}$$

and x is algebraically free over $H^*(\mathbf{S})^{\overline{\mathbf{G}}}$.

Proof. Identify $H^*(\overline{\mathbf{G}}\times\mathbf{S})$ with $H^*(\mathbf{S})[\,t\,]$ (note that we are using the same letter for both t and $\mathbf{p}_1^*(t)$), and let $\mu : H^*(\mathbf{S})[\,t\,] \longrightarrow H^*(\mathbf{S})$ be the homogeneous homomorphism which is the identity on $H^*(\mathbf{S})$ and which sends t to $-x$. Define $\varphi : H^*(\mathbf{S}) \longrightarrow H^*(\mathbf{S})$ as the composite $\varphi = \mu\circ\boldsymbol{\beta}^*$. Explicitly if $\boldsymbol{\beta}^*(\xi) = \sum_{i\geq 0}\xi_i t^i$, with $\xi_i\in H^*(\mathbf{S})$, then

$$\varphi(\xi) = \sum \xi_i(-x)^i \ .$$

We claim that the image of φ coincides with the ring of invariants. Since φ is clearly the identity on $H^*(\mathbf{S})^{\overline{\mathbf{G}}}$, it suffices to show that the image under φ of any ξ is invariant. We shall see below that this follows from the identity, in $H^*(\mathbf{S})[t,u]$,

$$\sum_{i\geq 0}\xi_i(t+u)^i = \sum_{i\geq 0}\ \boldsymbol{\beta}^*(\xi_i)u^i \ .$$

which, in this context, is a consequence of the associativity of the action together with the identity $\overline{\boldsymbol{\gamma}}^*(t) = t+u$ where $\overline{\boldsymbol{\gamma}}$ is the composition law on $\overline{\mathbf{G}}$. (Note that $\overline{\boldsymbol{\gamma}}^*(t)$ has to be a linear combination of t and u; an easy calculation shows that both coefficients are 1.)

Now, as promised, we have

$$\begin{aligned}\boldsymbol{\beta}^*(\varphi(\xi)) &= \sum_{i\geq 0}\boldsymbol{\beta}^*(\xi_i)\boldsymbol{\beta}^*(-x)^i = \\ &= \sum_{i\geq 0}\boldsymbol{\beta}^*(\xi_i)(-x-t)^i = \\ &= \sum \xi_i(-x)^i = \end{aligned}$$

In order to prove (9.1) it suffices to show that every homogeneous element ξ of $H^*(\mathbf{S})$ belongs to $R = H(\mathbf{S})^{\overline{\mathbf{G}}}[x]$. We proceed by induction on $\deg \xi$. We know that $\varphi(\xi) = \sum_{i \geq 0} \xi_i (-x)^i$ is invariant and, by induction hypothesis, that every ξ_i, $i > 0$, is already in R. It follows that ξ_0 also belongs to R. But $\xi_0 = \xi$, and hence $H^*(\mathbf{S})$ coincides with R (note that $\xi_0 = \xi$ because $s \mapsto (1_{\overline{\mathbf{G}}}, s)$ defines a section of $\boldsymbol{\beta}$.) Finally, since $\boldsymbol{\beta}^*(x) = x + t$ with t algebraically independent over $H^*(\mathbf{S})$, the element x is algebraically independent over the ring of invariants, and we are done.

Corollary 9.6. *In the situation of Proposition 9.3 one has*

$$H^*(\mathbf{S}) = H^*(\mathbf{S})^{\overline{\mathbf{G}}}[\, x\,] \; .$$

Remarks.

(1) In the situation of Proposition 9.5 the morphism $(\boldsymbol{\beta} \circ \boldsymbol{\eta}_{s_0})^*$, $s_0 \in \mathbf{S}$, sends x to t and coincides on $H^*(\mathbf{S})^{\overline{\mathbf{G}}}$ with the map induced in cohomology by the inclusion $\{s_0\} \hookrightarrow \mathbf{S}$.

(2) The corollary applies in particular to the Shatz strata $\mathbf{S}_P$ and gives a generalization of the isomorphism

$$H^*((\mathbf{Div}^{r,n}_{C/k})^{ss}) \simeq H^*(N(r,n))[\, x\,]$$

that was established in Section 7 under the assumption that r and n be coprime. In this case, the ring of invariants coincides with the image of $H^*(N(r,n)$ under $\boldsymbol{\vartheta}^*$.

(3) The assumption made in Proposition 9.3 about the prime ℓ can be relaxed but not eliminated; it is not difficult to describe an explicit set of generators of $H^2(\mathbf{Div}^{r,n}_{C/k})$, and to verify that the conclusion of the lemma holds if, and only if, ℓ is prime to the greatest common denominator of n and r.

10. Perfection of the Shatz stratification

Recall that for every effective divisor D we have the Shatz stratification

$$\mathrm{Div}^{r,n}_{C/k}(D) = \bigcup_{P \in \mathcal{P}_{r,n}} S_P(D) \; ,$$

where $S_P(D)$ is a locally closed subscheme, consisting of the divisors having Harder-Narasimhan filtration of type P.

If $\deg D$ is large enough, then $S_P(D)$ is smooth and has codimension d_P independent of D. As D varies the varieties $S_P(D)$ form an inductive system $\mathbf{S}_P$ and one has the stratification

$$\mathbf{Div}^{r,n}_{C/k} = \bigcup_{P \in \mathcal{P}_{r,n}} \mathbf{S}_P \; .$$

The set $\mathcal{P} = \mathcal{P}_{r,n}$ of Shatz (r,n)-polygons (i.e. strictly convex polygons in $\mathbb{R}^2$ joining $(0,0)$ with (r,n)) is partially ordered by the relation $P \leq P'$ if, and only if, P' lies above P. A subset I of $\mathcal{P}$ is open if $P \in I$ and $P \leq P'$ imply $P' \in I$. If I is open, then

is an open subset of $\mathrm{Div}^{r,n}_{C/k}(D)$ for each positive D (same statement for $\mathbf{S}_I = \bigcup_{P\in I}\mathbf{S}_P$.)

Let $I \subset \mathcal{P}$ be open and let P be a minimal element in the complement of I. The set $J = I\cup\{P\}$ is still open and, as a consequence of Proposition 4.1, $S_P(D)$ is closed in the open set $S_J(D)$.

Now if $D \le D'$, with $\deg D$ large enough, then we have a commutative diagram

$$\begin{array}{ccccccc}
\to & H^{i-2\,\mathrm{d}_P}(S_P(D')) & \longrightarrow & H^i(S_J(D')) & \longrightarrow & H^i(S_I(D')) & \to \\
 & \downarrow & & \downarrow & & \downarrow & \\
\to & H^{i-2\,\mathrm{d}_P}(S_P(D)) & \longrightarrow & H^i(S_J(D)) & \longrightarrow & H^i(S_I(D)) & \to
\end{array}$$

where the horizontal arrows are Gysin sequences. Using this diagram and the five-lemma one can show by induction on the cardinality of I that the cohomology of $H^*(\mathbf{S}_I)$, I finite, stabilizes. Moreover, taking the projective limit of these sequences we obtain the Gysin sequence

$$\to H^{i-2\,\mathrm{d}_P}(\mathbf{S}_P) \longrightarrow H^i(\mathbf{S}_J) \longrightarrow H^i(\mathbf{S}_I) \to \tag{10.1}$$

which is still exact.

Throughout the rest of this section, ℓ-adic cohomology will be $\mathbb{Q}_\ell$-cohomology. We shall say that the stratification is $\mathbb{Q}_\ell$-perfect, or perfect for short, if for every finite open subset I of $\mathcal{P}$ and for every one of the finitely many minimal elements λ of its complement, the Gysin sequence (10.1) splits into short exact sequences

$$0 \to H^{i-2\,\mathrm{d}_P}(\mathbf{S}_P) \longrightarrow H^i(\mathbf{S}_J) \longrightarrow H^i(\mathbf{S}_I) \to 0\ .$$

Recall from Proposition 5.1 that for every $d \ge 0$ the number of strata having codimension smaller than d is finite and that this number is independent of D, for $\deg D$ large enough.

Proposition 10.1. *The Shatz stratification of the ind-variety of (r,n)-divisors is perfect. In particular, there is an identity of Poincaré series*

$$P(\mathbf{Div}^{r,n}_{C/k};t) = \sum_{P\in\mathcal{P}_{r,n}} P(\mathbf{S}_P;t)\ t^{2\,\mathrm{d}_P}\ .$$

Proof. We use ideas inspired by [2]. As is well known, the composition of the Gysin map $H^{i-2\,\mathrm{d}_P}(S_P(D)) \longrightarrow H^i(S_J(D))$ with the restriction map $H^i(S_J(D)) \longrightarrow H^i(S_P(D))$ is multiplication by the top Chern class of the normal bundle $N_P(D)$ of $S_P(D)$ in $\mathrm{Div}^{r,n}_{C/k}(D)$. If $D \le D'$ and $\deg D$ is large enough, then the restriction of $N_P(D')$ to $S_P(D)$ is $N_P(D)$. Therefore the bundles $N_P(D)$, as D varies, define a vector bundle $\mathbf{N}_P$ over $\mathbf{S}_P$ (in fact, over a cofinal subsystem of $\mathbf{S}_P$.) If the top Chern class $e(\mathbf{N}_P)$ of $\mathbf{N}_P$ is a non zero divisor in $H^*(\mathbf{S}_P)$, then the Gysin sequence (10.2) will split into short exact sequences, as desired.

It suffices to prove, with the notation of Proposition 7.1, that $e(\boldsymbol{\delta}^*\mathbf{N}_p) = \boldsymbol{\delta}^*(e(\mathbf{N}_p))$ is not a zero divisor in $H^*((\mathbf{Div}^{r'_1,d'_1}_{C/k})^{ss}\times\cdots\times(\mathbf{Div}^{r'_l,d'_l}_{C/k})^{ss})$.

Let $\mathbf{G}_i$ and $\overline{\mathbf{G}}_i$ be the ind-varieties in monoids associated to $GL(r'_i,K)$ as in Section 9. For $i=1,\dots,l$ fix divisors $F_i \in (\mathbf{Div}^{r'_i,d'_i}_{C/k})^{ss}$ and consider the morphism

given by $\mathbf{j}\left((g_1,\ldots,g_l)\right)=(g_1(F_1),\ldots,g_l(F_l))$.

Using the Künneth formula and the results of Section 8 we can identify the cohomology ring of the right hand side of (10.2) with a polynomial ring $R^*[x_1,\ldots,x_l]$ where

$$R^* = \bigotimes_{1\le i\le l} H^*((\mathbf{Div}^{r'_i,d'_i}_{C/k})^{ss})^{\overline{\mathbf{G}}_i}$$

Also we have

$$H^*(\overline{\mathbf{G}}_1\times\cdots\times\overline{\mathbf{G}}_l)=\mathbb{Q}_\ell[t_1,\ldots,t_r]$$

where $t_1,\ldots,t_r$ are algebraically independent. In fact each t_i is the Chern class of the line bundle $\boldsymbol{\tau}_i$ obtained by pulling back to $\overline{\mathbf{G}}_1\times\cdots\times\overline{\mathbf{G}}_l$ the fundamental line bundle $\boldsymbol{\tau}$ over $\overline{\mathbf{G}}_i$. Note that the restriction of $\mathbf{j}^*$ to $R^*=\oplus_{i\ge 0}R^i$ is just projection onto $R^0=\mathbb{Q}_\ell$, and that $\mathbf{j}^*$ assigns t_i to each x_i.

Using this description of $\mathbf{j}^*$ it is immediately verified that a cohomology class α cannot be a zero divisor whenever $\mathbf{j}^*(\alpha)\neq 0$ (cf. §13 in [2].) Hence, in order to reach our goal we must show that

$$\mathbf{j}^*(e(\boldsymbol{\delta}^*\mathbf{N}_P))=e(\mathbf{j}^*\boldsymbol{\delta}^*\mathbf{N}_P)\neq 0$$

Recall from Section 4 that

$$\mathbf{N}_P=R^1\mathbf{q}_*\mathcal{H}om_+(\boldsymbol{\mathcal{U}}_P,\boldsymbol{\mathcal{U}}_P)$$

where $\boldsymbol{\mathcal{U}}_P$ is the universal family of divisors parametrized by $\mathbf{S}_P$, $\mathcal{H}om_+$ refers to the universal Harder-Narasimhan filtration of $\boldsymbol{\mathcal{U}}_P$ and $\mathbf{q}:C\times\mathbf{S}_P\longrightarrow\mathbf{S}_P$ is the natural projection.

Recall that if X is a scheme and $\mathcal{F}$ is an $\mathcal{O}_X$-module having a filtration

$$0\subseteq\mathcal{F}_1\subseteq\mathcal{F}_2\cdots\subseteq\mathcal{F}_l=\mathcal{F}$$

such that $\mathcal{F}/\mathcal{F}_1$ is locally free, then there is a short exact sequence

$$0\to\mathcal{H}om_+(\mathcal{F}/\mathcal{F}_1,\,\mathcal{F}/\mathcal{F}_1)\longrightarrow\mathcal{H}om_+(\mathcal{F},\mathcal{F})\longrightarrow\mathcal{H}om(\mathcal{F}_1,\mathcal{F}/\mathcal{F}_1)\to 0\ .$$

Using this fact, it is easily verified that $\mathcal{H}om_+(\boldsymbol{\mathcal{U}}_P,\boldsymbol{\mathcal{U}}_P)$ is flat over $\mathbf{S}_P$. We also have $H^2(\mathcal{H}om_+(\mathcal{U}_p(D),\,\mathcal{U}_p(D))|_{C\times\{E\}})=0$ for every $E\in S_P(D)$, and therefore base change commutes with taking $R^1\mathbf{q}_*$. Hence

$$\boldsymbol{\delta}^*(\mathbf{N}_P)=R^1\mathbf{q}'_*((1_C\times\boldsymbol{\delta})^*\mathcal{H}om_+(\boldsymbol{\mathcal{U}}_P,\boldsymbol{\mathcal{U}}_P))$$

where

$$\mathbf{q}':C\times\prod_{1\le i\le l}(\mathbf{Div}^{r'_i,d'_i}_{C/k})^{ss}\longrightarrow\prod_{1\le i\le l}(\mathbf{Div}^{r'_i,d'_i}_{C/k})^{ss}$$

is the natural projection.

Let $\boldsymbol{\mathcal{U}}_i$ be the pull-back to $C\times\prod_{1\le i\le l}(\mathbf{Div}^{r'_i,d'_i}_{C/k})^{ss}$ of the universal family of divisors over $C\times(\mathbf{Div}^{r'_i,d'_i}_{C/k})^{ss}$. Then we have

$$(1_C \times \boldsymbol{\delta})^* \mathcal{H}om_-(\boldsymbol{\mathcal{U}}_P, \boldsymbol{\mathcal{U}}_P) = \bigoplus_{i \geq j} \mathcal{H}om(\boldsymbol{\mathcal{U}}_i, \boldsymbol{\mathcal{U}}_j)$$

and

$$\begin{aligned}(1_C \times \boldsymbol{\delta})^* \mathcal{H}om_+(\boldsymbol{\mathcal{U}}_P, \boldsymbol{\mathcal{U}}_P) &= \bigoplus_{i < j} \mathcal{H}om(\boldsymbol{\mathcal{U}}_i, \boldsymbol{\mathcal{U}}_j) \\ &= \bigoplus_{i < j} \boldsymbol{\mathcal{U}}_i^\vee \otimes \boldsymbol{\mathcal{U}}_j \ .\end{aligned}$$

It follows that

$$\boldsymbol{\delta}^* \mathbf{N}_P = \bigoplus_{i<j} R^1 \mathbf{q}'_* \boldsymbol{\mathcal{U}}_i^\vee \otimes \boldsymbol{\mathcal{U}}_j \ .$$

Since $\mathbf{j}^*$ commutes with $R^1\mathbf{q}'_*$, using Lemma 9.2, we obtain

$$\mathbf{j}^* \boldsymbol{\delta}^* \mathbf{N}_P = \bigoplus_{i<j} R^1 \mathbf{p}_{2*}(\mathbf{p}_1^*(F_i)^\vee \otimes \mathbf{p}_2^*(\boldsymbol{\tau}_i)^\vee \otimes \mathbf{p}_1^*(F_j) \otimes \mathbf{p}_2^*(\boldsymbol{\tau}_j))$$

where the F_i's are as in (10.2) and

$$C \xleftarrow{\mathbf{p}_1} C \times \overline{\mathbf{G}}_1 \times \cdots \times \overline{\mathbf{G}}_l \xrightarrow{\mathbf{p}_2} \overline{\mathbf{G}}_1 \times \cdots \times \overline{\mathbf{G}}_l$$

are the natural projections. Now, using the projection formula, we obtain

$$\mathbf{j}^* \boldsymbol{\delta}^* \mathbf{N}_P = \bigoplus_{i<j} (\boldsymbol{\tau}_i^\vee \otimes \boldsymbol{\tau}_j)^{h^1(F_i^\vee \otimes F_j)} \ .$$

and, therefore,

$$e(\mathbf{j}^* \boldsymbol{\delta}^* \mathbf{N}_P) = \prod_{i<j} (t_j - t_i)^{h^1(F_i^\vee \otimes F_j)} \neq 0$$

as required.

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